Queen Mary School Hainan
Queen Mary University of London

QHP5701 Exploratory Data Analysis

Periodicity & Fourier

Nikesh Bajaj, PhD
Lecturer in Data Science,
Queen Mary University of London
nikesh.bajaj@qmul.ac.uk
https://nikeshbajaj.in

#Ref: Chapter 1, 3, 4, Oppenheim

Contents

QHP5701 Exploratory Data Analysis

Periodicity

Nikesh Bajaj, PhD
Lecturer in Data Science,
Queen Mary University of London
nikesh.bajaj@qmul.ac.uk
https://nikeshbajaj.in

Note: Most of slides will be empty for in-class computations

Since we are following a text-book heavily, slides, will not include all the mathematical computations and details. Please check the text-book, for details.

#Ref: Chapter 1, Oppenheim

Periodicity

- Condition: $x(t) = x(t + T)$

$$x(t) = e^{j\omega_o t}$$

Periodicity

$$x(t) = x_1(t) + x_2(t) + ..$$

$$x(t) = 1 + 2\cos(10t + 1) - sin(4t - 1)$$

Periodicity

- Condition: $x(n) = x(n + N)$

$$x(n) = e^{j\Omega_o n}$$

$$e^{j\Omega_o(n+N)} = e^{j\Omega_o n}$$

$$e^{j(\Omega_o+k2\pi)n} = e^{j\Omega_o n}$$

Periodicity

$$x(n) = x_1(n) + x_2(n) + ..$$

$$x(n) = e^{j\frac{2\pi}{3}n} + \mathrm{e}^{j\frac{3\pi}{4}n}$$

Periodicity

- Harmonically Related
  - Fundamental frequency f_0
  - mth harmonic $f_m = mf_0$

$$x(t) = \cos(10\pi t) + cos(20\pi t)$$

Online Demos

Sinusoidal Player :

- https://nikeshbajaj.github.io/teaching/demos/SP/sinusoidal_player.html

Oscillator

- https://c4fa.github.io/nikJS/SineWave.html

Fourier Series

- https://www.falstad.com/fourier/

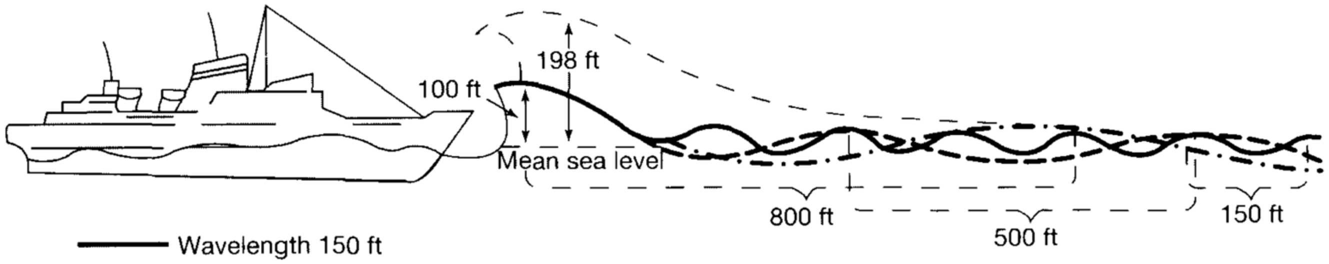
Contents

- Periodicity
- Fourier
  - Fourier Series **#Ref: Chapter 3, Oppenheim**
  - Fourier Transform **#Ref: Chapter 4, Oppenheim**

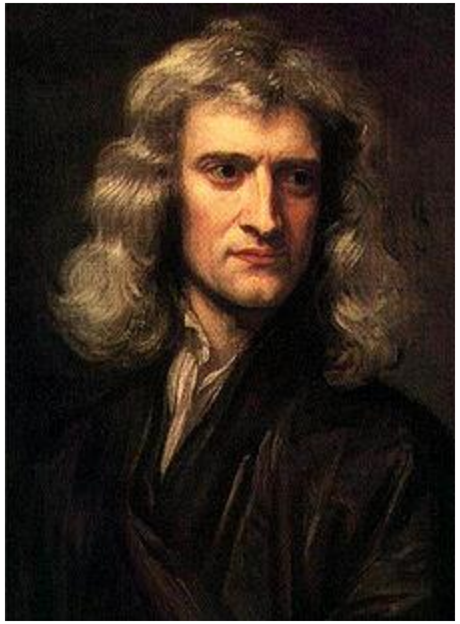
QHP5701 Exploratory Data Analysis

Fourier: Analysis and Synthesis Tool

Nikesh Bajaj, PhD
Lecturer in Data Science,
Queen Mary University of London
nikesh.bajaj@qmul.ac.uk
https://nikeshbajaj.in



Guess....?



(4 January 1643 – 31 March 1727)

(1749-1827)

(1768 –1830)

History

In 1748

- **L. Euler**: Vibrating String
  - Vertical deflation at any time is the linear combination of ‘normal modes’, he showed it

In 1753

- **D. Bernouli** argued same on physical ground (No math) but was not accepted.
- idea of ‘Trigonometric series’ was discarded

In 1759

- **J. L. Langrange** strongly criticized Trig. Series.

"Fourier Series is a Great Mathematical Poem"
- Lord Kelvin (William Thompson)

Fourier

Jean Baptiste Joseph Fourier
21 March 1768 – 16 May 1830, Auxerre, France
Mathematician, Egyptologist, Revolutionary Discovery (1822). Scientific Advisor
Work: heat conduction

- He claimed that any periodic signal can be represented by a series of harmonically related sinusoidal.

FOURIER SERIES

- He also obtained a representation of Aperiodic signal, not as weighted sum of harmonically related sinusoidal, but as weighted Integral of sinusoidal that are not at all harmonically related.

FOURIER INTEGRAL or TRANSFORM

Contents

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#Ref: Chapter 3, Oppenheim

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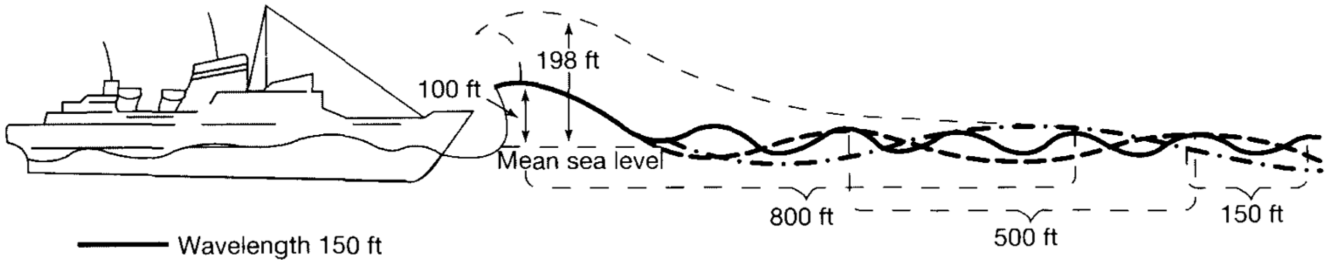
Fourier Series



Nikesh Bajaj, PhD

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Any periodic signal can be represented by a series of harmonically related sinusoidal.

Fourier Series

Fourier Series

$$fund.period\ T_0 = \frac{2\pi}{\omega_0}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

- Synthesis equation

$$\mathrm{a_k} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} dt$$

- Analysis equation

$$\mathrm{a_k} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\frac{2\pi}{T_0}t} dt$$

$$\mathrm{a_{-k}} = \mathrm{a_k^*}$$

Fourier Series

$x(t) = \cos(3\pi t/4)$

$x(t) = \sin(3\pi t/4)$

Fourier Series

$x(t) = 1 + 2\cos(2\pi t) + \sin(3\pi t)$

Fourier Series

$$x(t) = \begin{cases} 1 & for\ |t| < T_1 \\ 0 & for\ |t| > T_1 \end{cases} \quad with\ period\ T_0$$

$$x(t) = \begin{cases} 1 \ \ for\ |t| < T_1 \\ 0 \ \ for\ |t| > T_1 \end{cases} \qquad with\ period\ T_0$$

Fourier Series

$$x(t) = \begin{cases} 1 \;\; for\ |t| < T_1 \\ 0 \;\; for\ |t| > T_1 \end{cases} \qquad with\ period\ T_0$$

Fourier Series

$T_0 = 4T_1$

$$x(t) = \begin{cases} 1 \;\; for\ |t| < T_1 \\ 0 \;\; for\ |t| > T_1 \end{cases} \quad with\ period\ T_0$$

Fourier Series

$T_0 = 8T_1$

$$x(t) = \begin{cases} 1 \ \ for\ |t| < T_1 \\ 0 \ \ for\ |t| > T_1 \end{cases} \qquad with\ period\ T_0$$

Fourier Series

$T_0 = 16T_1$

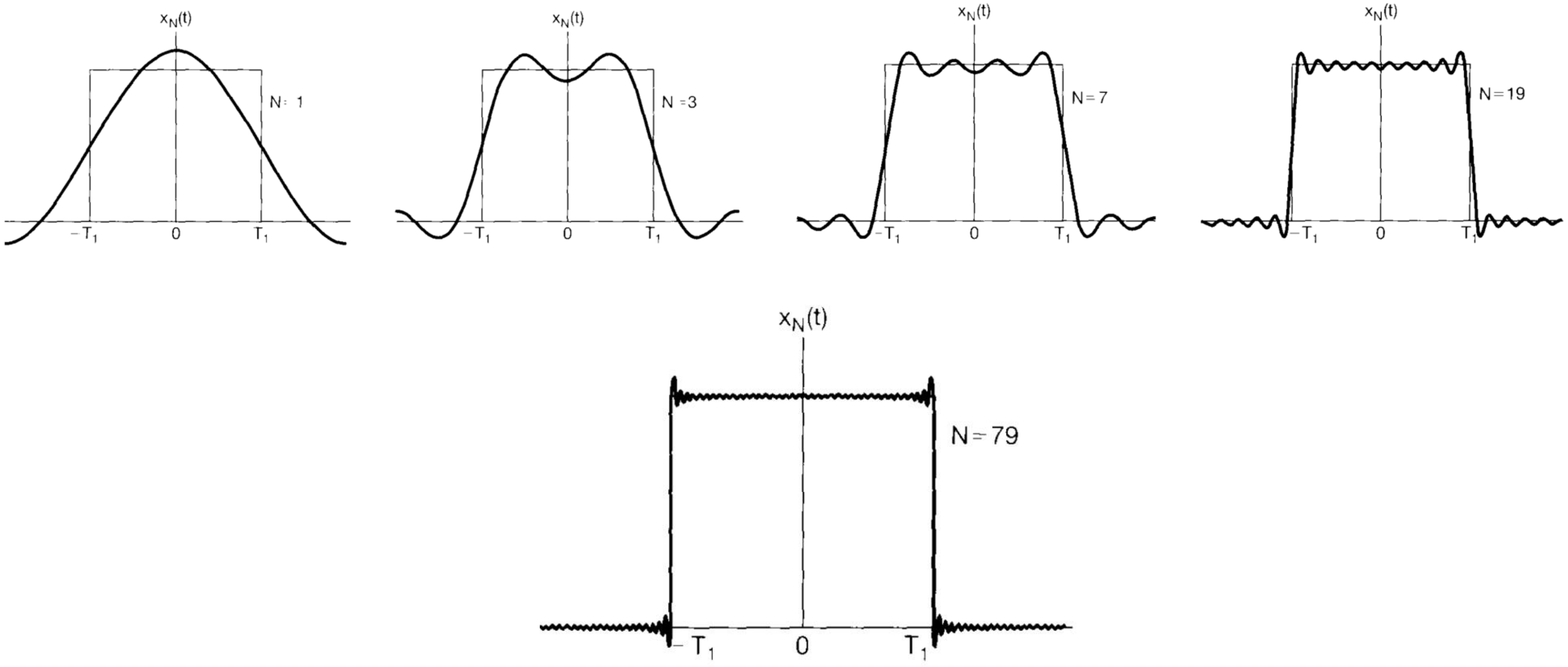
$$x(t) = \begin{cases} 1 & for\ |t| < T_1 \\ 0 & for\ |t| > T_1 \end{cases} \quad with\ period\ T_0$$

Fourier Series

- In general, $a_k's$ are complex
- Spectrum: Magnitude and Phase

Fourier Series: Synthesis

Synthesis with finite number of components



[Demo: https://www.falstad.com/fourier/](https://www.falstad.com/fourier/)

Fourier Series: Convergence

Dirichlet condition

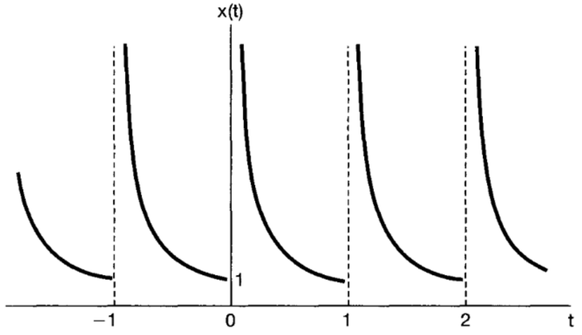
Developed by P. L. Dirichlet

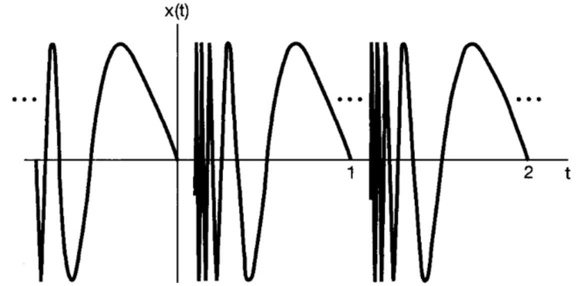
- <u>Condition 1</u>: *Over any period* T, $x(t)$ *must be absolutely integrable; that is*

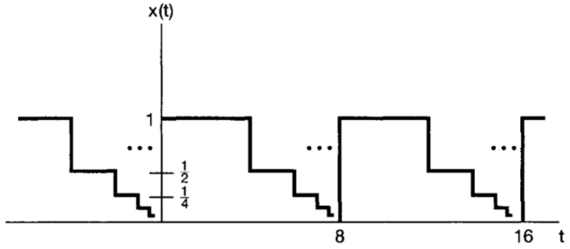
$$\int_T |x(t)|\, dt < \infty$$

- <u>Condition 2</u>: *In any finite interval of time,* $x(t)$ *is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.*

- <u>Condition 3:</u> *In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.*







Fourier Series: Properties & Pairs

- Linearity
- Time- shifting
- Time Reversal
- Time-scaling
- Multiplication
- Parseval's Relation

Table from Book

Chapter: 3

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$x_e(t) = \mathcal{Ev}\{x(t)\}$ [$x(t)$ real]; $x_o(t) = \mathcal{Od}\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{a_k\}$; $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Contents

- Periodicity
- Fourier
  - Fourier Series
  - Fourier Transform

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#Ref: Chapter 4, Oppenheim

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Fourier Transform

Nikesh Bajaj, PhD

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Any aperiodic signal can be represented as weighted Integral of sinusoidal that are not at all harmonically related.

Fourier Transform

Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

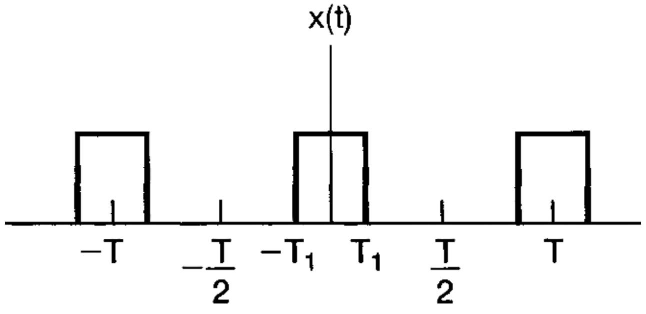
- Synthesis equation

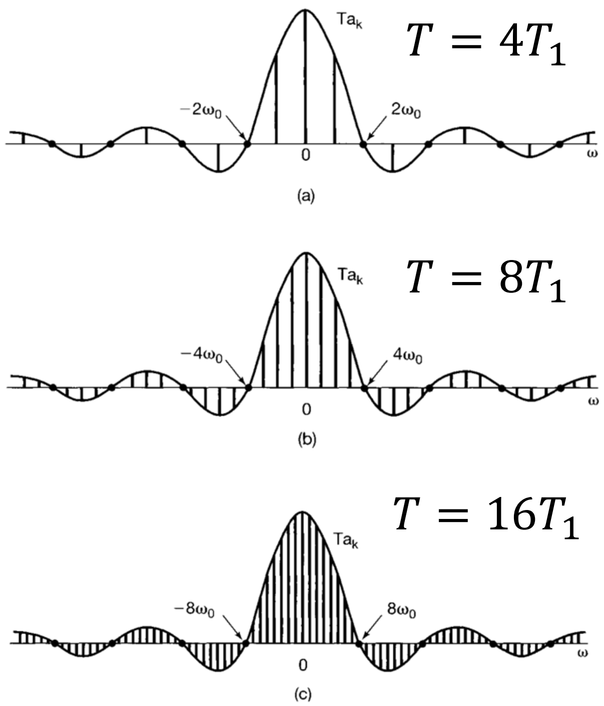
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}dt$$

- Analysis equation

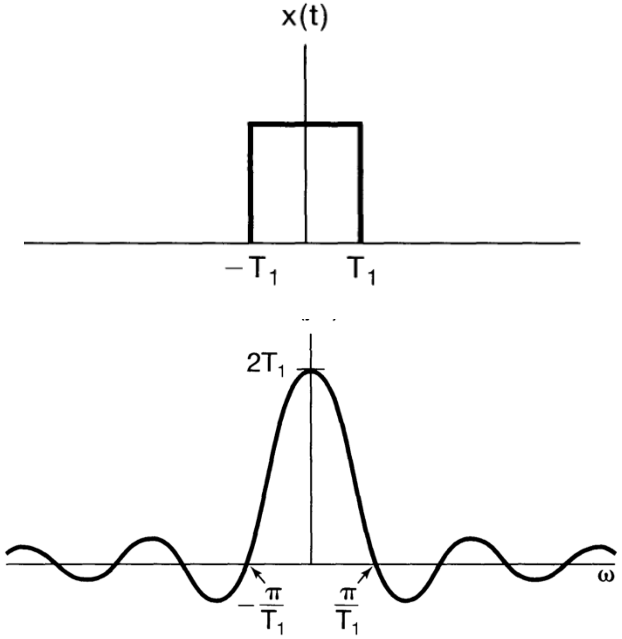
Fourier Transform

Fourier Series to Fourier Transform





$T \to \infty$



Fourier Transform

$x(t) = e^{-at}u(t)$ a>0

Fourier Transform

$x(t) = e^{-a|t|}$ a>0

Fourier Transform

$x(t) = \delta(t)$

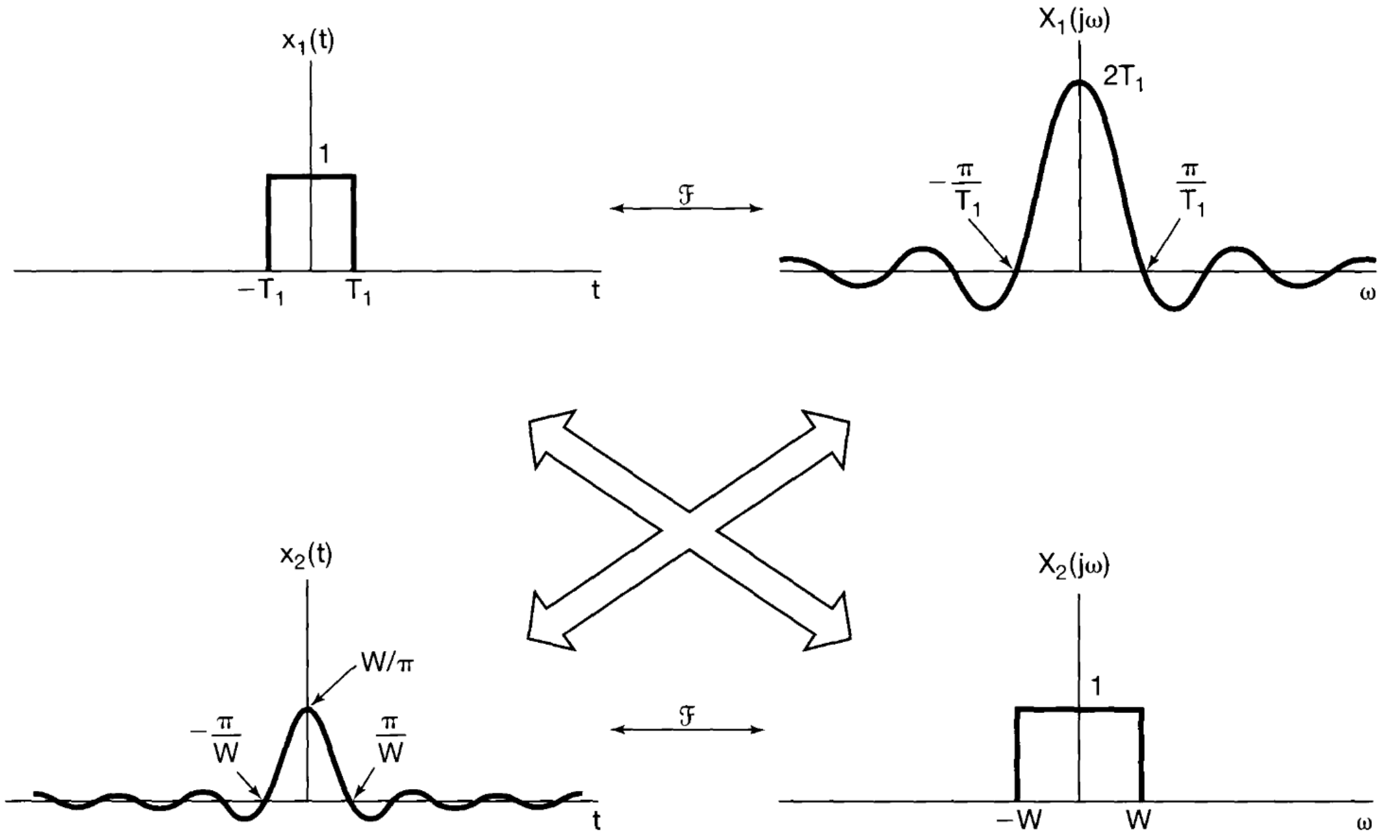
Fourier Transform

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

Fourier Transform: Inverse

$$X(j\omega) = \begin{cases} 1, & |\omega| < W_1 \\ 0, & |\omega| > W_1 \end{cases}$$

Fourier Transform: Duality



Fourier Transform: Convergence

Dirichlet condition

- Condition 1: $x(t)$ *must be absolutely integrable; that is*

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

- Condition 2: $x(t)$ *have finite number of maxima and minima for any finite interval of time.*

- Condition 3: $x(t)$ *have a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.*

FT Properties

Chapter: 4

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt} x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega} X(j\omega)$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

FT Pairs

Chapter: 4

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$x(t) = 1$	$2\pi\,\delta(\omega)$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$

Fourier Transform: Properties

Linearity

$x(t) = e^{-at}u(t) + 3e^{-bt}u(t)$ a, b>0

Fourier Transform: Properties

Time-shift

$x(t) = e^{-a(t+1)}u(t+1)$ a>0

Fourier Transform: Properties

Frequency-shift

$$X(j\omega) = \begin{cases} 1, & W_1 < |\omega| < 2W_1 \\ 0, & else. \end{cases}$$

Online Demos

Sinusoidal Player :

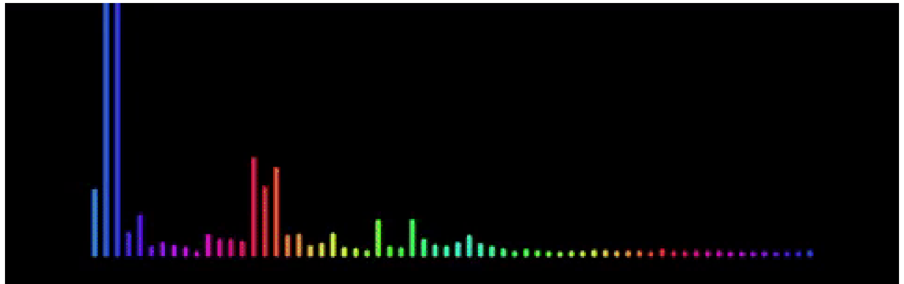
- https://nikeshbajaj.github.io/teaching/demos/SP/sinusoidal_player.html

Oscillator

- https://c4fa.github.io/nikJS/SineWave.html

Fourier Series

- https://www.falstad.com/fourier/
- https://www.desmos.com/calculator



Exercises: Do at home

Homework

Book: Alan V. Oppenheim

Chapter 3 : Fourier Series – only continues-time

Chapter 4: Fourier Transform – only continues-time

Examples

Basic Problems

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