Queen Mary School Hainan
Queen Mary University of London

QHP5701 Exploratory Data Analysis

Filtering & **Sampling**

Nikesh Bajaj, PhD
Lecturer in Data Science,
Queen Mary University of London
nikesh.bajaj@qmul.ac.uk
https://nikeshbajaj.in

#Ref: Chapter 1, 3, 4, Oppenheim

Contents

- Filtering – python demo
- Sampling
- Sampling Theorem

#Ref: Chapter 7, Oppenheim

Convolution and Fourier

$$y(t) = x(t) * h(t)$$
$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$y(t) = x(t)h(t)$$
$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * H(j\omega)$$

Filtering

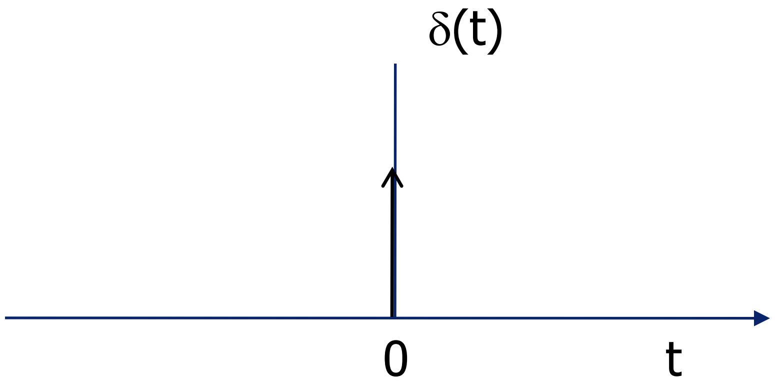
Python – Demo

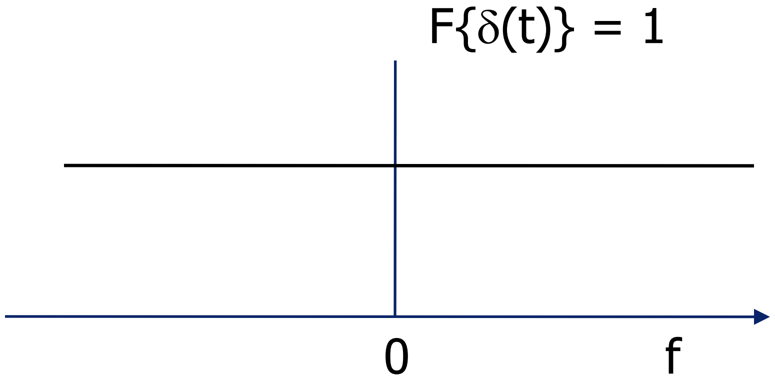
DFT Analysis

Signal and Spectrum - Fourier transform

Fourier Transform

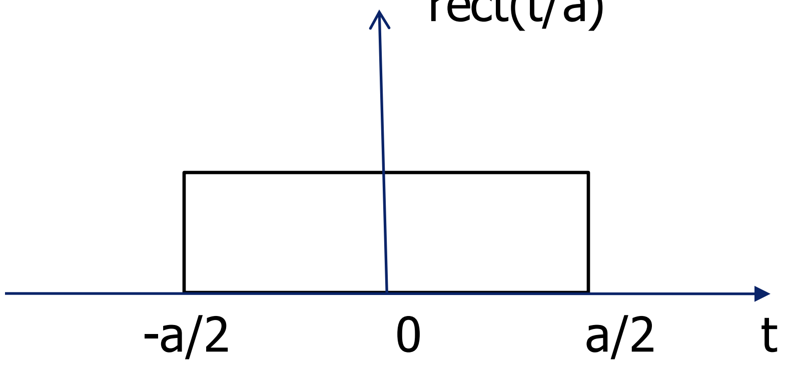
- Impulse Response $\delta(t)$

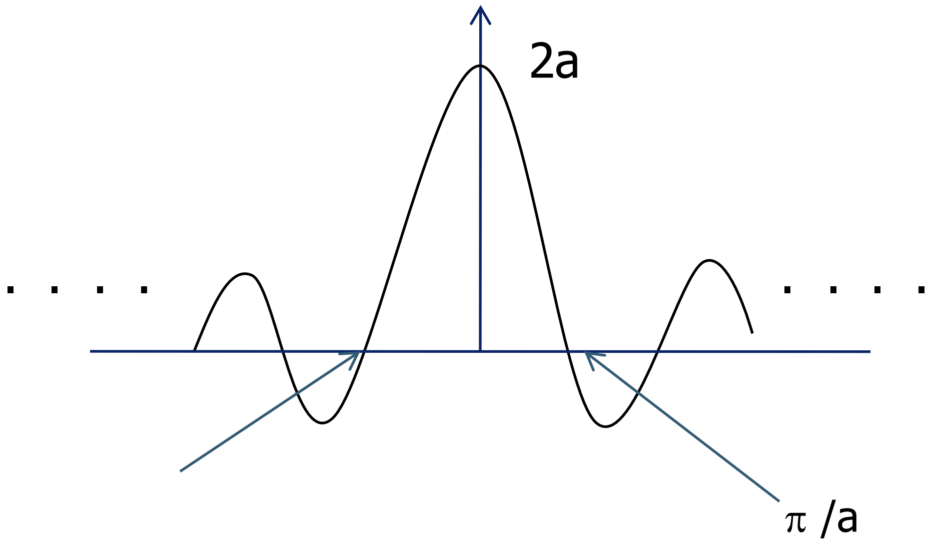




Fourier Transform

- Gate Function $G_a(t)$: rect(t/a)

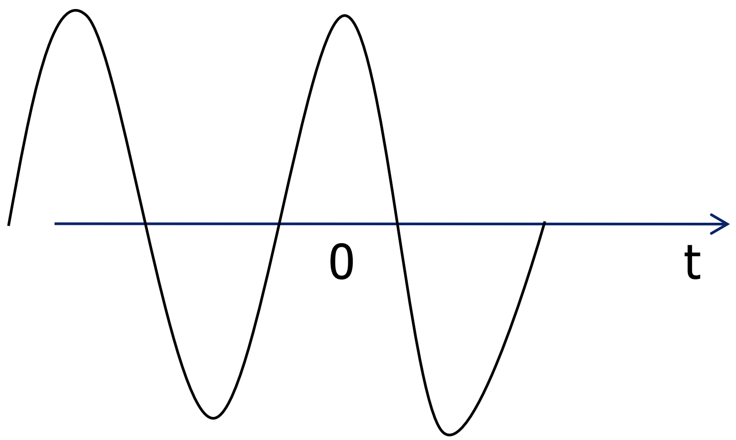


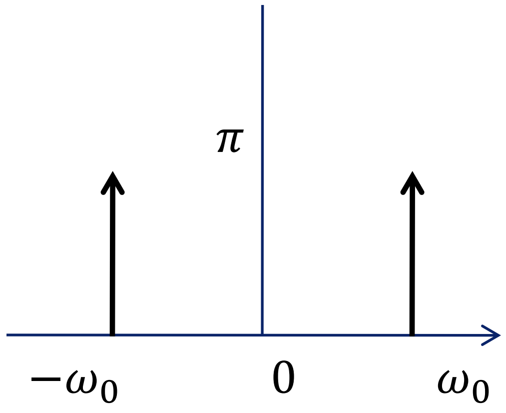


Fourier Transform

Cosine signal

$\cos(\omega_0 t)$





Pulse Train p(t)

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$P(j\omega) = ?$$

Sampling

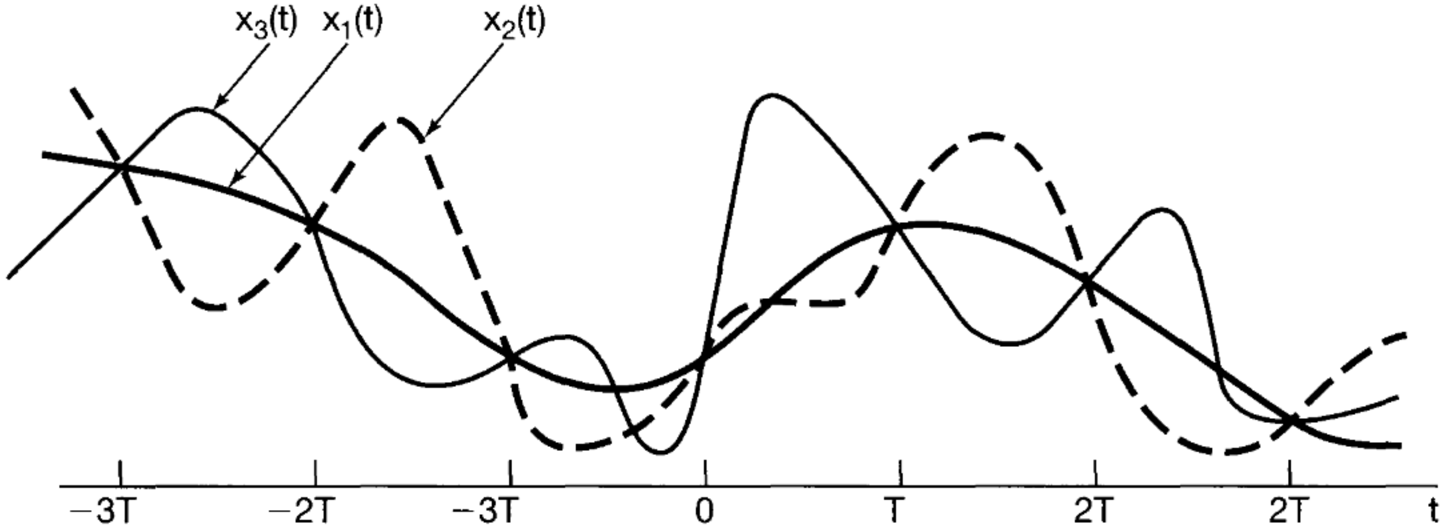
A process of converting continuous-time signal x(t) to discreet time sequence x(n)

$$x(n) = x(nT_s),$$

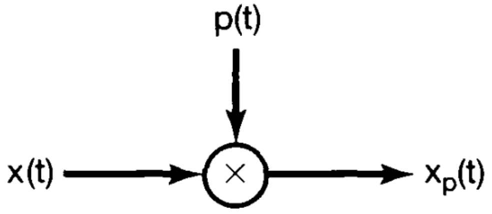
$$for\ samping\ period\ T_s$$

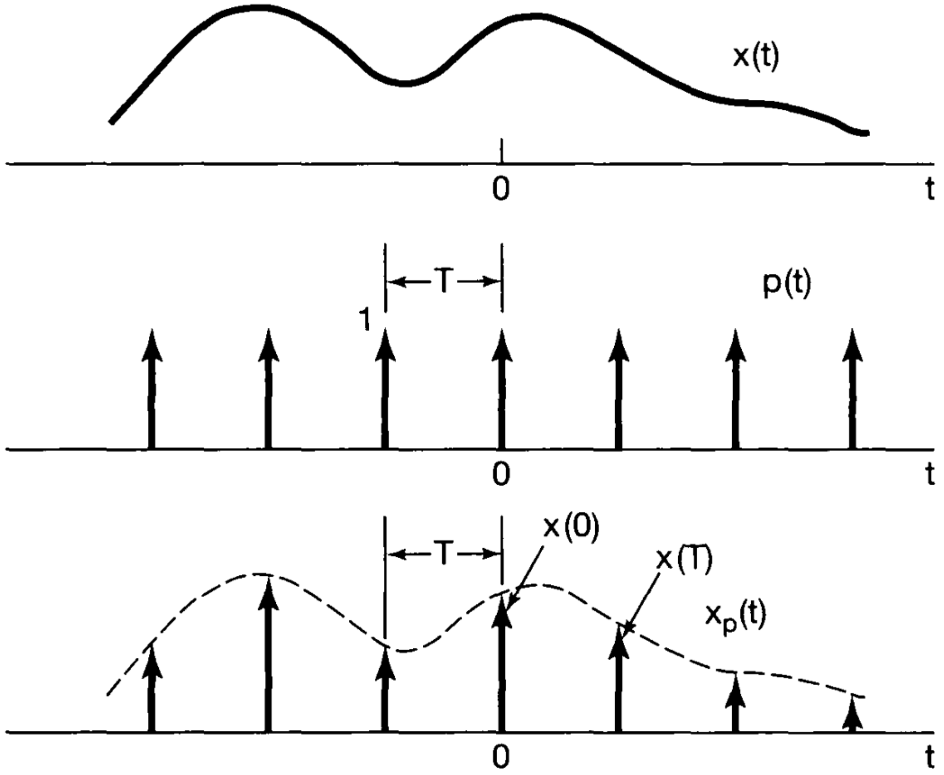
$$Sampling\ Frequency\ f_s = \frac{1}{T_s}$$

Sampling



Impulse Train Sampling





Sampling Theorem: $\omega_s > 2\omega_h$

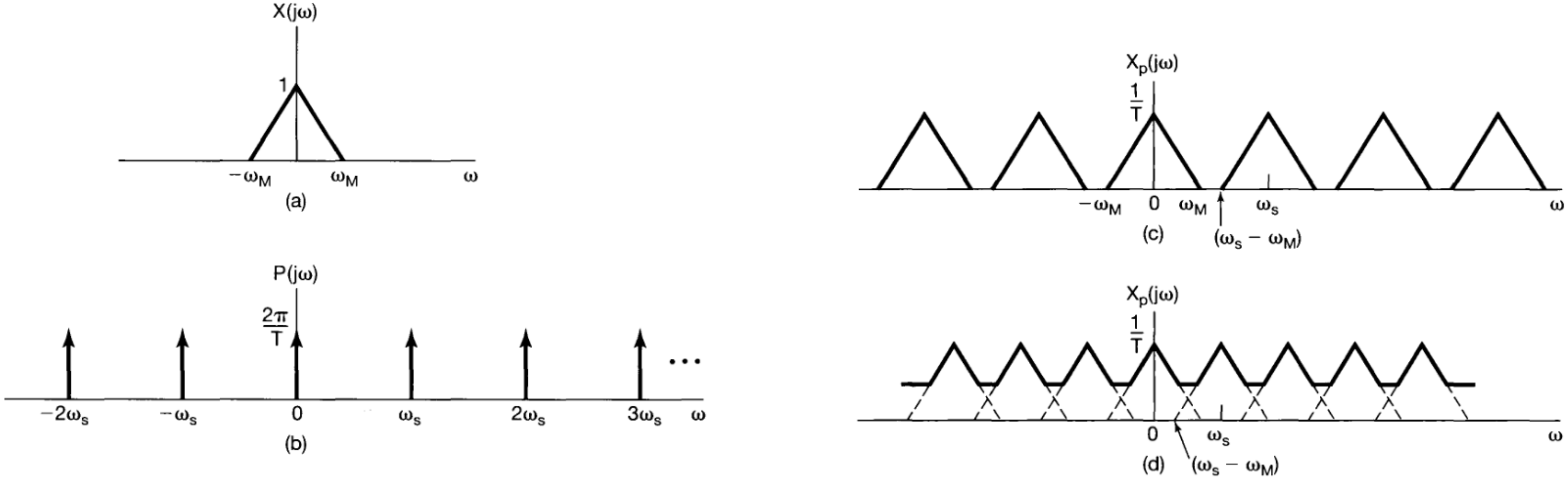
x(t) - a band limited signal

$$y(t) = x(t)p(t)$$

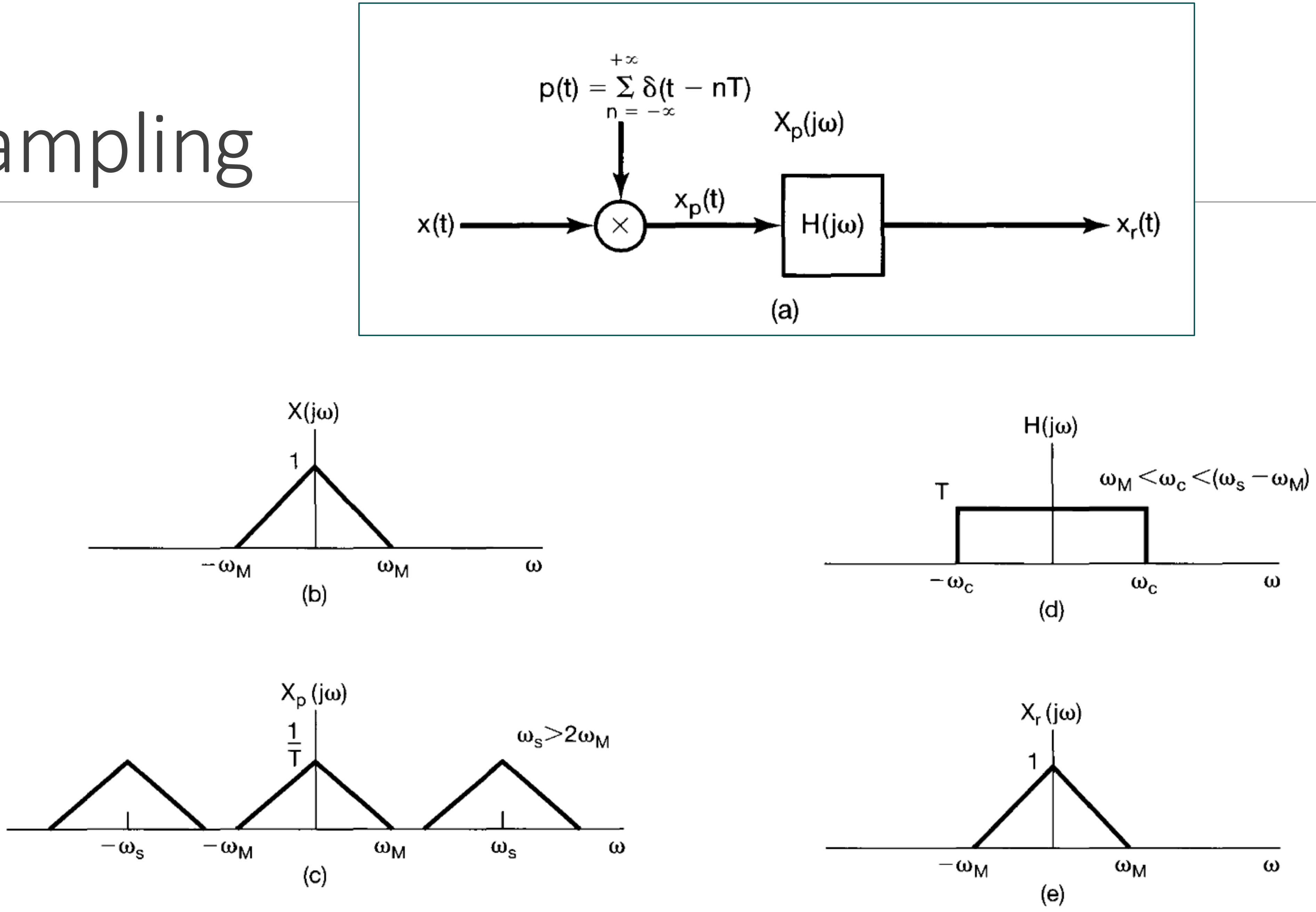
$$Y(j\omega) =?$$

Sampling Theorem

Sampling Theorem: Aliasing



Sampling



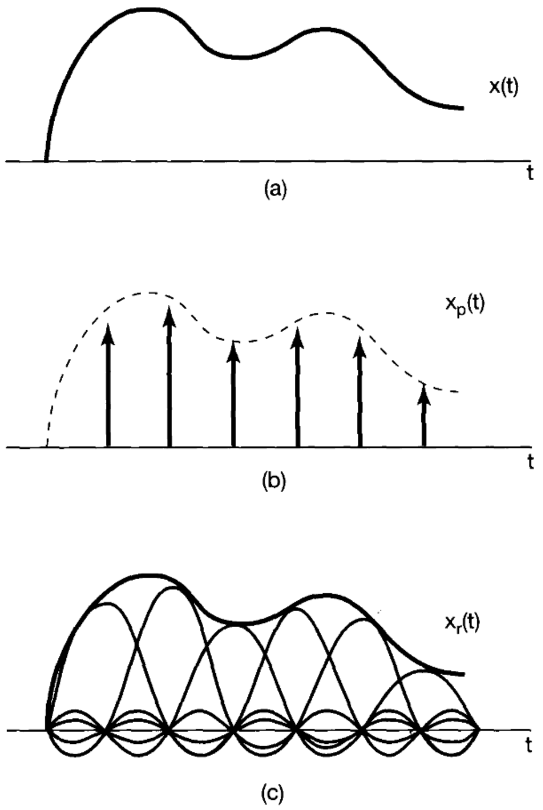
p(t) = Σ δ(t − nT), n = −∞ to +∞
x(t)
$x_p(t)$
$X_p(j\omega)$
H(jω)
$x_r(t)$
(a)
X(jω)
1
$-\omega_M$
ω_M
ω
(b)
$X_p(j\omega)$
1/T
$\omega_s > 2\omega_M$
$-\omega_s$
$-\omega_M$
ω_M
ω_s
ω
(c)
H(jω)
T
$\omega_M < \omega_c < (\omega_s - \omega_M)$
$-\omega_c$
ω_c
ω
(d)
$X_r(j\omega)$
1
$-\omega_M$
ω_M
ω
(e)

Lowpass filter

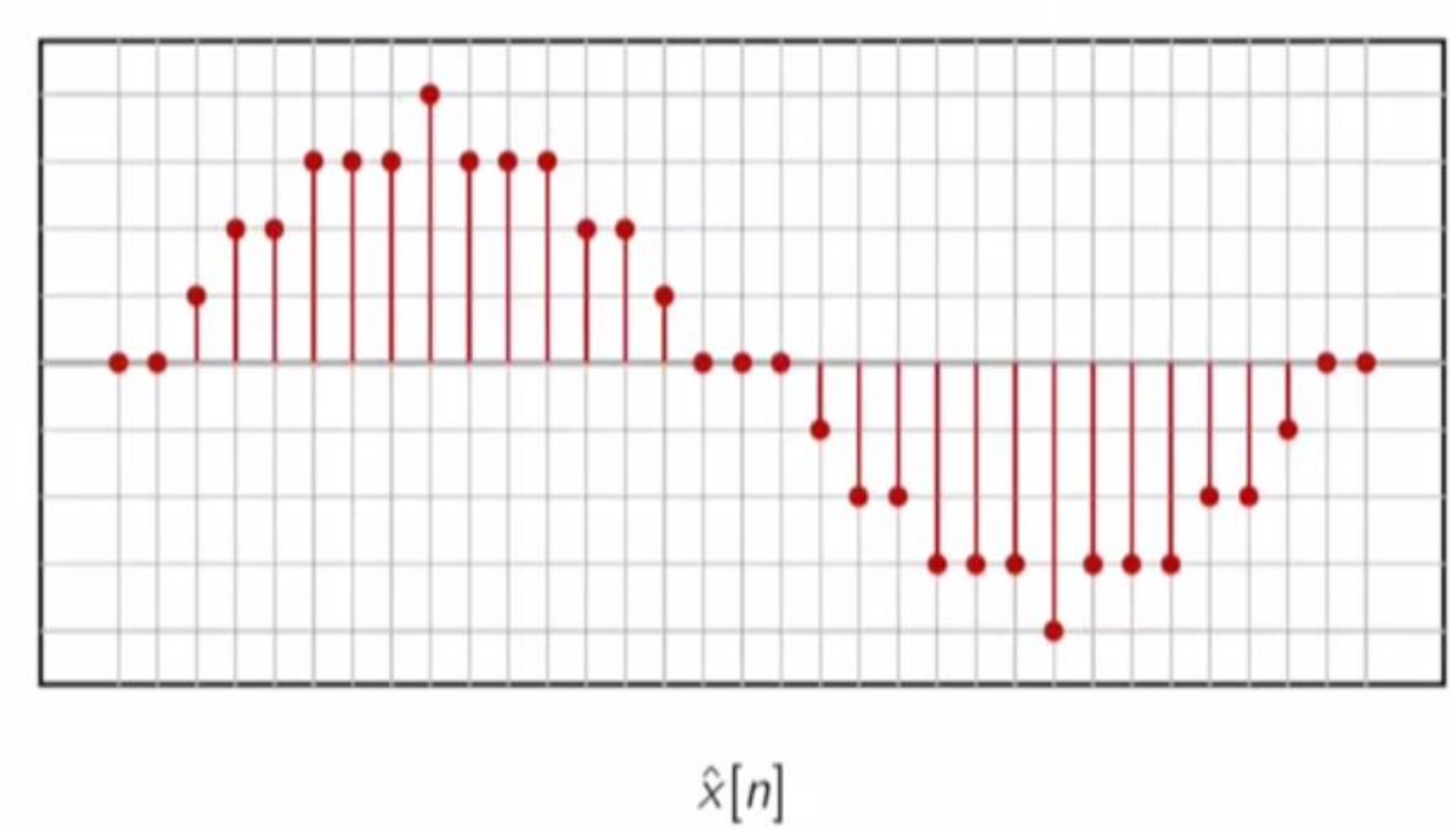
Ideal Lowpass filter

Constraints on cut-off frequency

$$\omega_h < \omega < \omega_s - \omega_h$$



Sampling + Quantization



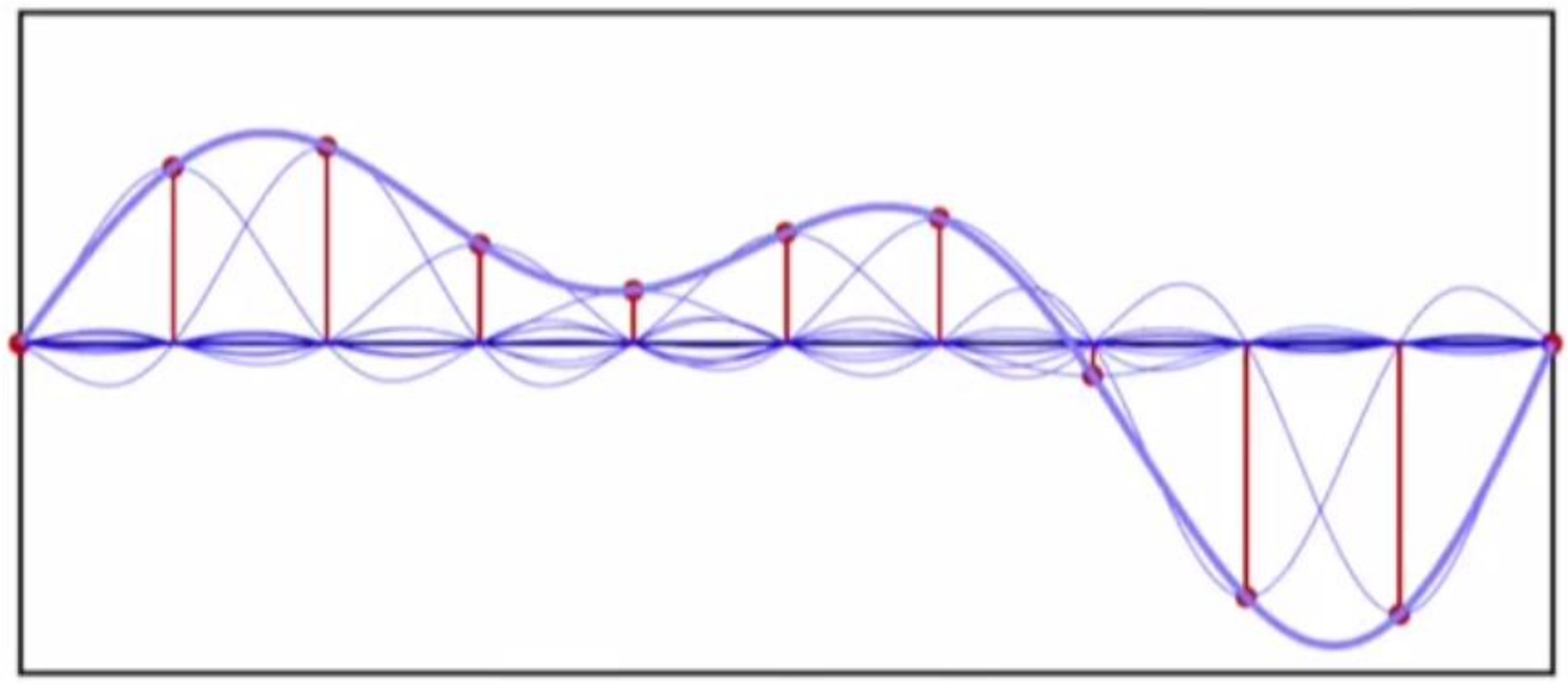
By: Nikesh
Bajaj

Sampling Theorem

Under appropriate "slowness" conditions for $x(t)$ we have:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t - nT_s)/T_s)}{\pi(t - nT_s)/T_s}$$

Reconstruction of Signal



Limitation

1) Ideal delta function
2) $\omega_h \to \infty$
3) Ideal lowpass filter

Sampling Frequency: Examples

Sampling frequency:

- Speech
- Audio, music
- EEG
- ECG
- ..

$Direc\ Delta\ \delta(t)$

$$\delta(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{t^2 + \epsilon}$$

$$\delta(t) = \frac{1}{2} \lim_{\epsilon \to 0} \epsilon |t|^{\epsilon - 1}$$

$$\delta(t) = \lim_{\sigma \to 0} \frac{1}{\sqrt{4\pi\sigma}} e^{-t^2/4\sigma}$$ Gaussian Function

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\pi t} \sin(t/\epsilon)$$ Sin Function

$$x(t) = 1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega)$$

$$x(t) = 1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega)$$

Useful:

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} \quad \text{for } |\alpha| < 1$$

$$\sum_{k=0}^{M} \alpha^k = \frac{1-\alpha^{M+1}}{1-\alpha} \quad \text{for } |\alpha| < 1$$

Systems: Exercise

1.17. Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by

$$y(t) = x(\sin(t)).$$

(a) Is this system causal?
(b) Is this system linear?

1.18. Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k],$$

where n_0 is a finite positive integer.

(a) Is this system linear?
(a) Is this system time-invariant?
(c) If $x[n]$ is known to be bounded by a finite integer B (i.e., $|x[n]| < B$ for all n), it can be shown that $y[n]$ is bounded by a finite number C. We conclude that the given system is stable. Express C in terms of B and n_0.

Systems: Exercise

1.19. For each of the following input-output relationships, determine whether the corresponding system is linear, time invariant or both.

(a) $y(t) = t^2 x(t-1)$ **(b)** $y[n] = x^2[n-2]$

(c) $y[n] = x[n+1] - x[n-1]$ **(d)** $y[n] = \mathcal{O}d\{x(t)\}$

1.20. A continuous-time linear system S with input $x(t)$ and output $y(t)$ yields the following input-output pairs:

$$x(t) = e^{j2t} \xrightarrow{S} y(t) = e^{j3t},$$

$$x(t) = e^{-j2t} \xrightarrow{S} y(t) = e^{-j3t}.$$

(a) If $x_1(t) = \cos(2t)$, determine the corresponding output $y_1(t)$ for system S.

(b) If $x_2(t) = \cos(2(t - \frac{1}{2}))$, determine the corresponding output $y_2(t)$ for system S.

Systems: Exercise

(1) Memoryless
(2) Time invariant
(3) Linear
(4) Causal
(5) Stable

(a) $y(t) = x(t-2) + x(2-t)$
(b) $y(t) = [\cos(3t)]x(t)$
(c) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$
(d) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$
(e) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$
(f) $y(t) = x(t/3)$
(g) $y(t) = \frac{dx(t)}{dt}$

1.28. Determine which of the properties listed in Problem 1.27 hold and which do not hold for each of the following discrete-time systems. Justify your answers. In each example, $y[n]$ denotes the system output and $x[n]$ is the system input.

(a) $y[n] = x[-n]$
(b) $y[n] = x[n-2] - 2x[n-8]$
(c) $y[n] = nx[n]$
(d) $y[n] = \mathcal{E}v\{x[n-1]\}$
(e) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$
(f) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$
(g) $y[n] = x[4n+1]$

Systems: Exercise

1.30. Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals to the system that have the same output.

(a) $y(t) = x(t-4)$

(b) $y(t) = \cos[x(t)]$

(c) $y[n] = nx[n]$

(d) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

(e) $y[n] = \begin{cases} x[n-1], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

(f) $y[n] = x[n]x[n-1]$

(g) $y[n] = x[1-n]$

(h) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau)d\tau$

(i) $y[n] = \sum_{k=-\infty}^{n}(\frac{1}{2})^{n-k}x[k]$

(j) $y(t) = \frac{dx(t)}{dt}$

(k) $y[n] = \begin{cases} x[n+1], & n \geq 0 \\ x[n], & n \leq -1 \end{cases}$

(l) $y(t) = x(2t)$

(m) $y[n] = x[2n]$

(n) $y[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

Convolution: Exercise

2.1. Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \quad \text{and} \quad h[n] = 2\delta[n+1] + 2\delta[n-1].$$

Compute and plot each of the following convolutions:

(a) $y_1[n] = x[n] * h[n]$ **(b)** $y_2[n] = x[n+2] * h[n]$

(c) $y_3[n] = x[n] * h[n+2]$

2.8. Determine and sketch the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \leq t \leq 1 \\ 2-t, & 1 < t \leq 2 \\ 0, & \text{elsewhere} \end{cases},$$

$$h(t) = \delta(t+2) + 2\delta(t+1).$$

Convolution: Exercise

2.2. Consider the signal

$$h[n] = \left(\frac{1}{2}\right)^{n-1} \{u[n+3] - u[n-10]\}.$$

Express A and B in terms of n so that the following equation holds:

$$h[n-k] = \begin{cases} (\frac{1}{2})^{n-k-1}, & A \leq k \leq B \\ 0, & \text{elsewhere} \end{cases}.$$

2.3. Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2],$$

$$h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.

2.4. Compute and plot $y[n] = x[n] * h[n]$, where

$$x[n] = \begin{cases} 1, & 3 \leq n \leq 8 \\ 0, & \text{otherwise} \end{cases},$$

$$h[n] = \begin{cases} 1, & 4 \leq n \leq 15 \\ 0, & \text{otherwise} \end{cases}.$$

Convolution: Exercise

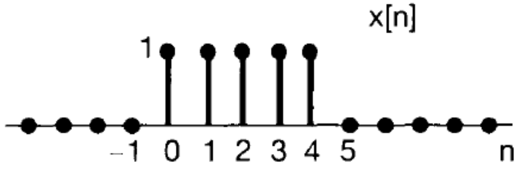
2.21. Compute the convolution $y[n] = x[n] * h[n]$ of the following pairs of signals:

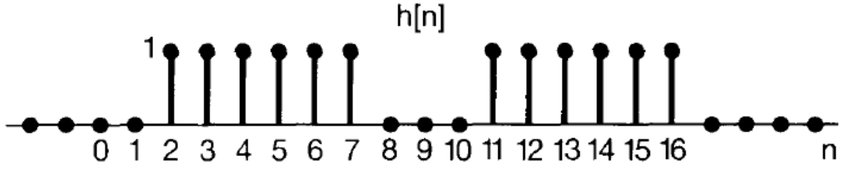
(a) $\left.\begin{aligned} x[n] &= \alpha^n u[n], \\ h[n] &= \beta^n u[n], \end{aligned}\right\} \quad \alpha \neq \beta$

(b) $x[n] = h[n] = \alpha^n u[n]$

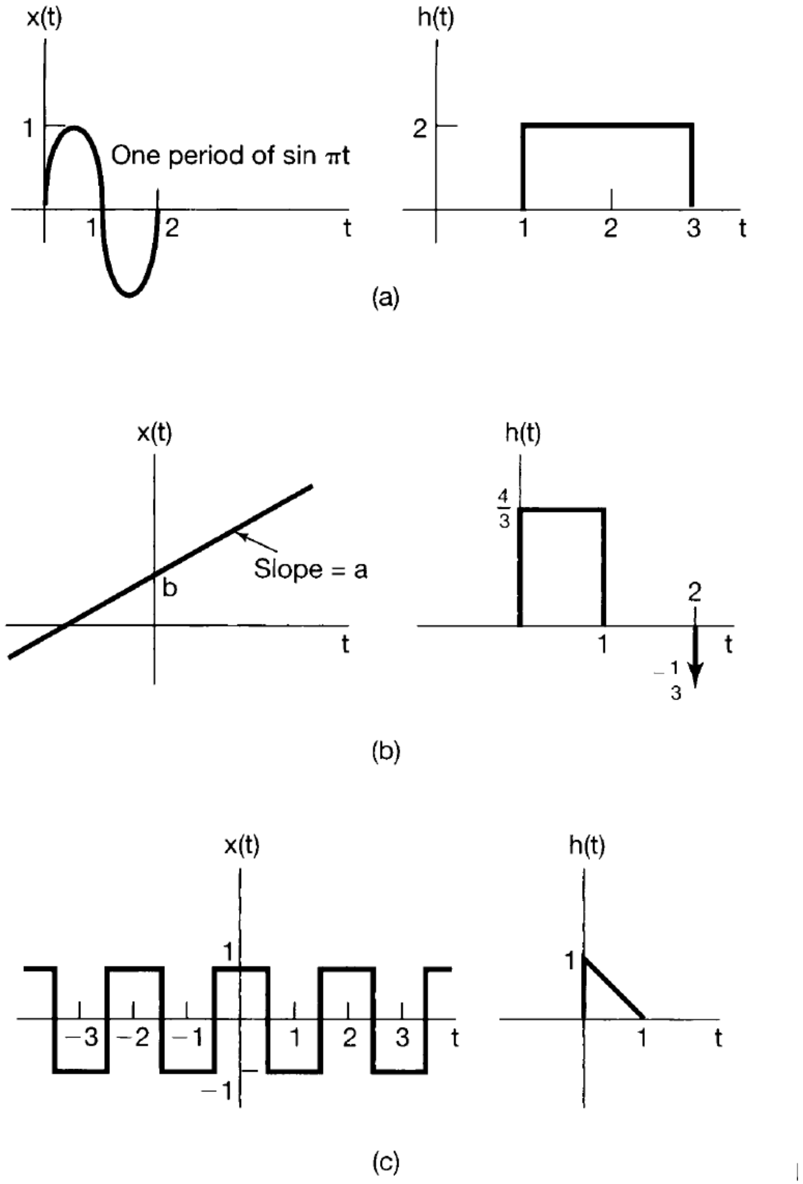
(c) $x[n] = (-\frac{1}{2})^n u[n-4]$

$h[n] = 4^n u[2 - n]$





Convolution: Exercise



Exercises: Do at home

Homework

Book: Alan V. Oppenheim

Chapter 1 : System Properties

Chapter 2: Convolution

Examples

Basic Problems

Queen Mary
University of London