A little more about Fourier

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

$$x(t) = 1 \xrightarrow{FT} 2\pi\delta(\omega) = X(j\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int \cancel{2\pi}\,\delta(\omega)e^{j\omega t}\,d\omega$$

$$= \int \delta(\omega)e^{j\omega t}\,d\omega = 1$$

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QHP5701 Exploratory Data Analysis

Time Series Analysis

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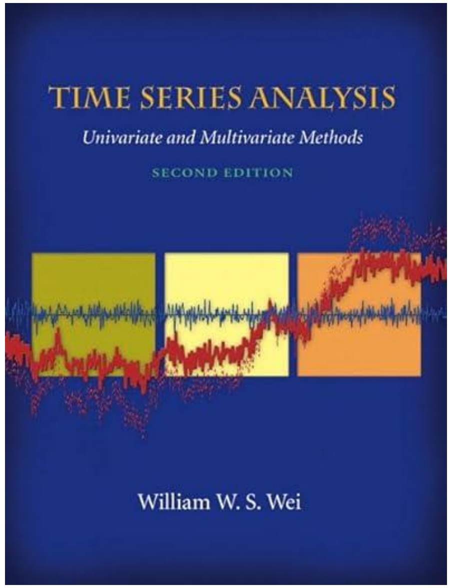
#Ref: Chapter 1 to 4, +
Time Series Analysis by William WS Wei

Time Series Analysis

Time Series Analysis
Sources & Processing

- Stochastics processes
- *Correlation*
- ACF, AVF, PACF, Noise, Visualisation
- MA, AR, ARMA, ARIMA
- Stationary & Non-stationary
- Seasonality

- Time Series Analysis



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QHP5701 Exploratory Data Analysis

Stochastic Process

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#Ref: Chapter 2, TSA, William WS Wei

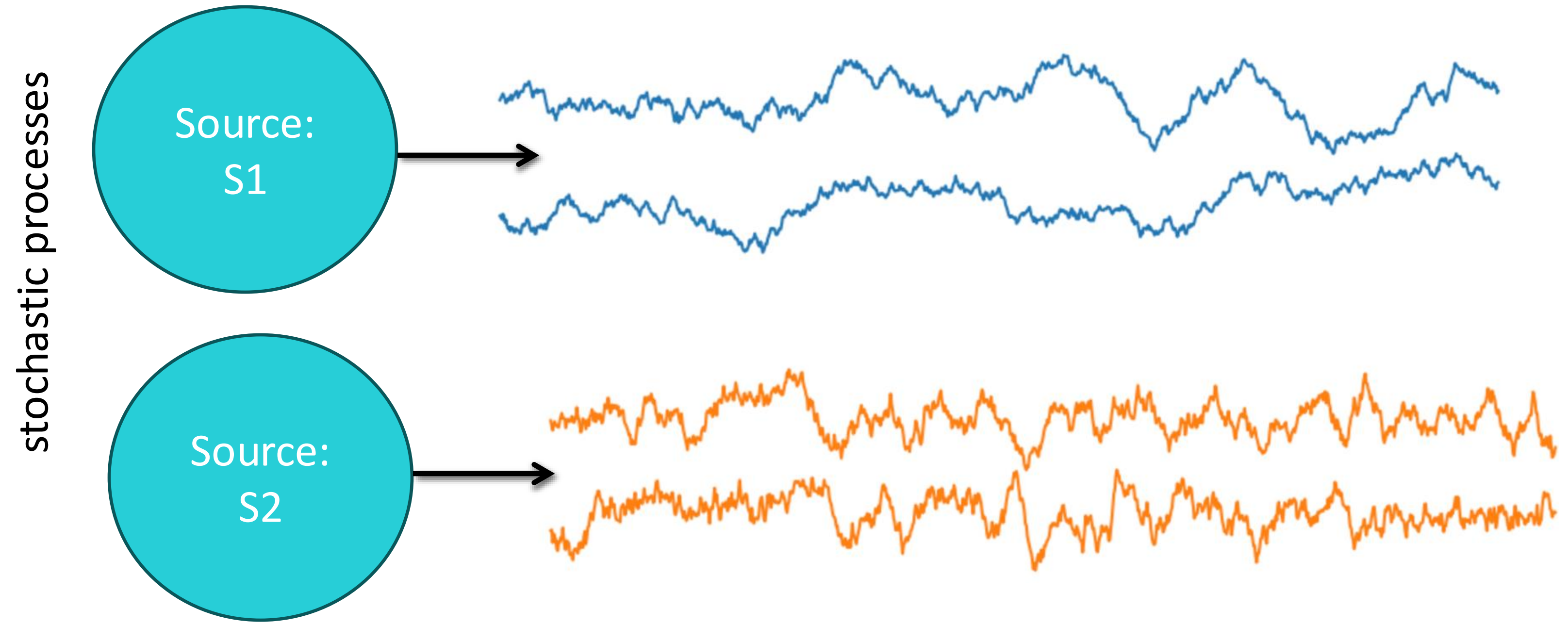
Time Series Analysis: Source

Popul → Sample

$x(t)$ $t \in \mathbb{Z}$

- Time Series - temporal discreet data - time-dependent - ***signal x(n)***
- All time-series are considered to be generated by **a source**
- Objective of TSA is to analyse the source, from given a time-series

"***Time series is a realization of a stochastic process***"

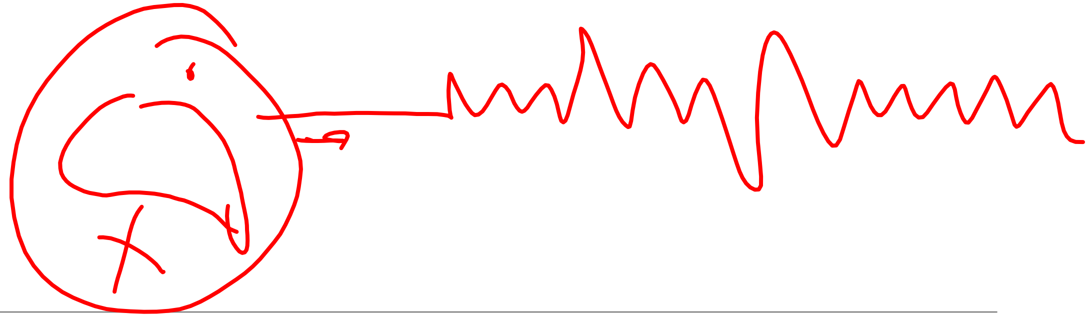


Time Series Analysis: Source

- We analyse time-series, in order to know **more about the source**

- We will model these sources using some mathematical models (stochastic processes)
  - AR (q), MA(p), ARMA, ARIMA

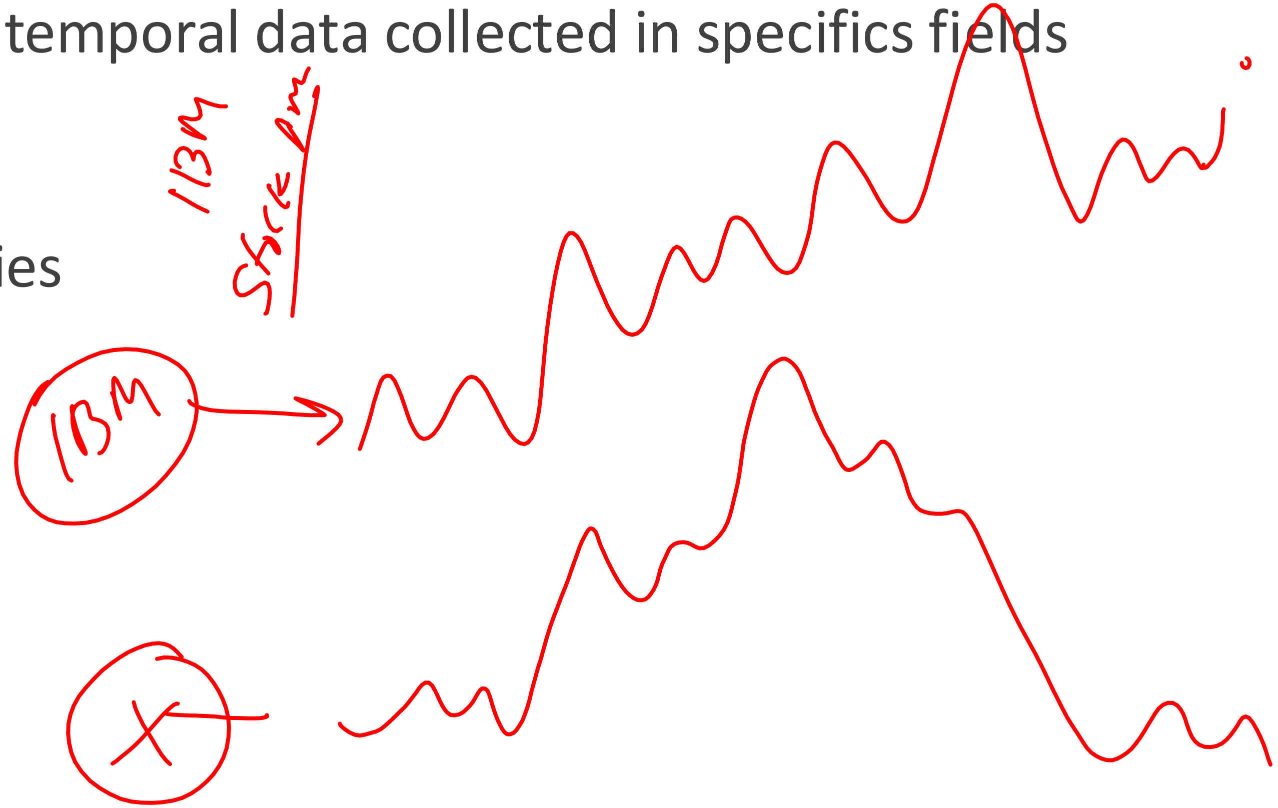
- We will analyse properties of time-series, that help to understand, which model they come from
  - ACF, AVF, PACF



Time Analysis

Time series are typically referred to temporal data collected in specifics fields such as:

- Marketing time series
- Economics and Financial time series
- Demographic time series
- Population time series
- Physical time series



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QHP5701 Exploratory Data Analysis

Stochastic Process: Fundamentals

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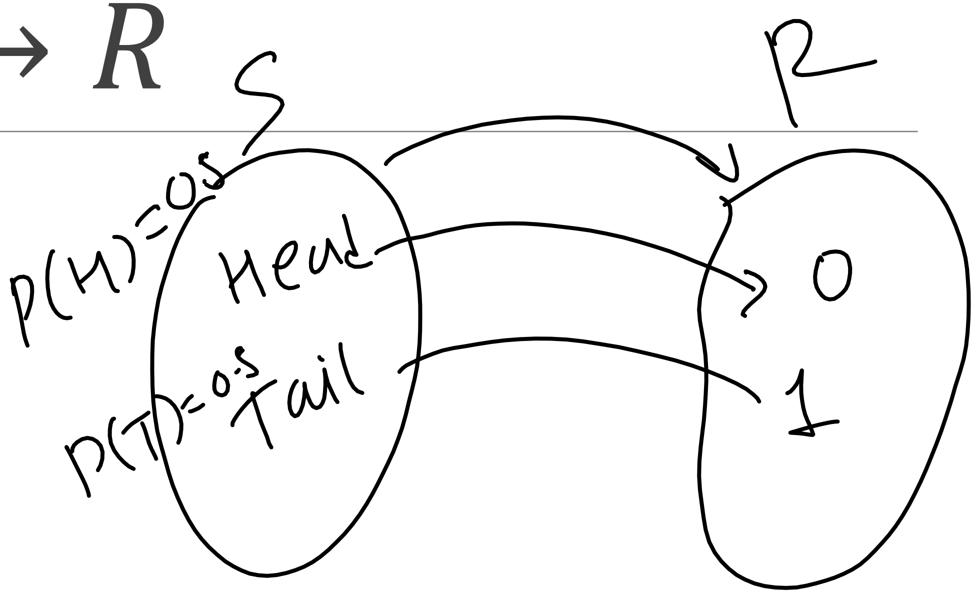
#Ref: Chapter 2, TSA, William WS Wei

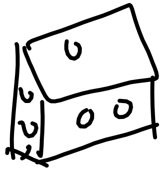
Random Variable $X: S \to R$

Random variable

- Not quite a variable x from algebra
- Ways to map random processes to numbers
- $X: S \to R$

- Continues/ Discreet random Variable





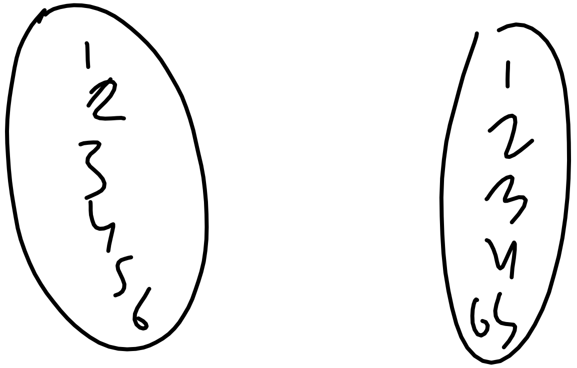
Examples

- $X = \begin{cases} 1, & for\ head \\ 0, & for\ tail \end{cases}$

$$\frac{P(X = 1)}{P(X=0)}$$

- $Y = sum\ of\ two\ dice$

$P(Y < 3)$



$P(X=4) = 1/6$

Discreet RV

$S : \{H, T\}$

$\{0, 1\}$

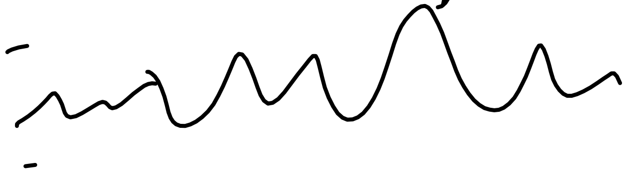
$P(H) = 0.5$

$P(T) = 0.5$

Cond RV.

$S : (-1, 1)$

$x(n) =$

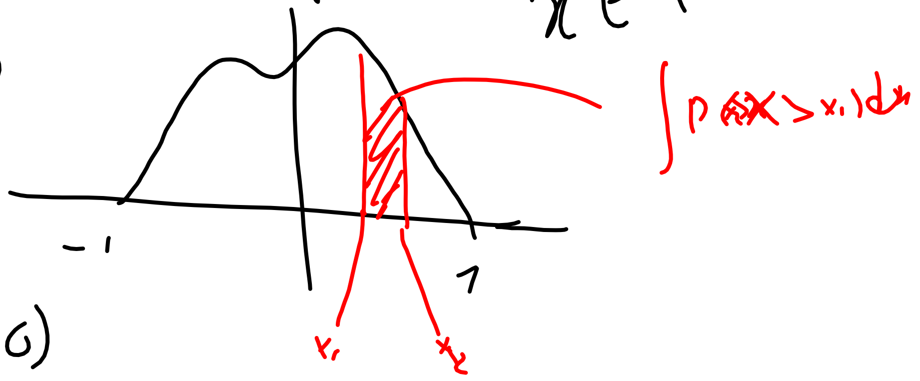


0 0.0001

0 12345

$x \in R$

$P(x)$



$P(x=0)$

$P(X \geq 0)$

Expected value

$E(X)$

$X \rightarrow 0\ 0\ 1\ 1\ 0\ 1\ 1$

Expected value $E[X]$ is a mean of random variable X

X

$$\mu_X = E[X] = \sum x p(X = x)$$

$$\int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$\mu_X = E[X] \equiv \text{mean of source X}$$

$E(X)$

Expected value: Example

Head / Tail | Coin Toss

$X = \begin{cases} H = 0 \\ T = 1 \end{cases}$

$\cancel{P(H)}$ $p(X=H) = 0.5$

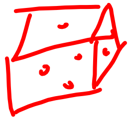
$p(X=T) = 0.5$

$X \rightarrow 0,1,1,0,1,0$ $E[X] = \sum x\, p(X=x) = \mu_X = 0.5$

$= 0 \times p(0) + 1\, p(1)$

$= 0 + 0.5 = 0.5$

Expected value: Example

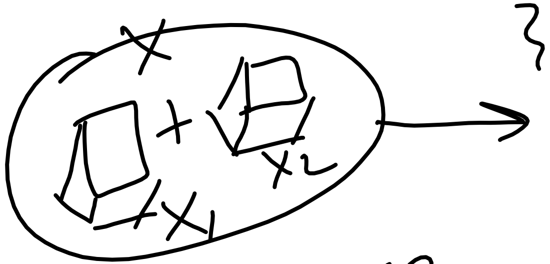
Dice  $X \rightarrow 1, 2, 1, 1, 6, 3, 4, 5,$

$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases} \qquad p(X=x) = 1/6$$

$$E(X) = \mu_X = \sum x\,p(x) = 1/6 + 2/6 + 3/6 + 4/6 + 5/6 + 6/6$$

$$= 21/6 = 7/2 = 3.5$$

$$\mu_X = 3.5$$



$\{2.$

123

Dice + Dice

$\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

$E(X) = \cancel{6}\ \cancel{5}\ 7 = \mu_X$

$p(x)$

$X = \begin{cases} 2 \text{ — } 1/36 \\ 3 \text{ — } 2+1,\ 1+2 \quad 2/36 \\ 4 \text{ — } 3/36 \text{ — } \{1+3,\ 3+1,\ 2+2,\ \cancel{2+2} \\ 5 \text{ — } 4/36 \text{ — } 1+4,\ 2+3,\ 3+2,\ 4+1 \\ 6 \text{ — } 5/36 \text{ — } 1+5,\ 2+4,\ 3+3,\ 4+2,\ 5+1 \\ 7 \rightarrow 6/36 \end{cases}$

$12 \rightarrow 1/36$

Estimation of Expected value: mean

Expected value $E[X]$ is a mean of random variable X

$$\mu_X = E[X] = \sum xp(X = x)$$

- Which is estimated from realization x, as sample mean,
  We don't know the TRUE probability distribution P(X)

$$\boxed{\hat{\mu}_X = \frac{1}{N}\sum x_i}$$

X

1
2
3
4
5
6
7

$\hat{\mu}_X$

Example

$X \longrightarrow 1, 2, 3, 4, 1, 1, 1, 3, 4, 5,$

iid

$E[X] = ?$

$\hat{\mu}_X = 25/10 = 2.5$

Example

$$X \sim N(0,1) \quad + \quad Y \sim N(0,1) \quad = \quad Z$$

$X \sim N(0,1)$

$E[X] = 0$

$Y \sim N(0,1)$

$E[Y] = 0$

$$E(Z) = E[Z] = E[X+Y]$$
$$= E[X] + E[Y]$$
$$= 0$$

$$E[(X-\mu_X)^2] = E(X^2) - E(\mu_X^2)$$

Variance & Standard Deviation

Variance: $V[X] = \underline{E[(X-\mu_X)^2]} = \sigma_X^2$ $= \cancel{\sum (x-\mu_X)^2 P(X=x)}$

Standard Deviation = σ_X $= \cancel{\sum (x^2+\mu_X^2-2\mu_X x)\, P(X=x)}$

$$E[(X-\mu_X)^2] = E[(X-\mu_X)(X-\mu_X)]$$

$$=$$

$$= \sum (x-\mu_X)^2 P(X=x)$$

Estimation of Variance & Standard Deviation

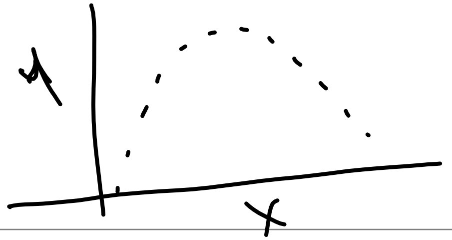
Variance: $V[X] = E[(X-\mu_X)^2] = \sigma_X^2$

$Standard\ Deviation = \sigma_X$

- This is estimated from realization x,

$$\boxed{\hat{\sigma}_X^2 = \frac{1}{N-1}\sum(x_i - \hat{\mu}_X)^2}$$

Covariance



Variance of x

$$\sigma_X^2 = E[(X-\mu_X)^2] = E[(X-\mu_X)(X-\mu_X)] = \sum (x-\mu_X)^2 P(X=x) = V(X)$$

Covariance of X and Y

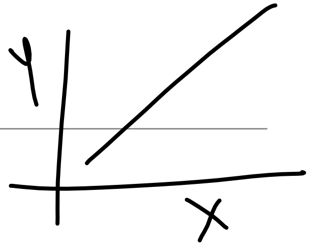
$$\sigma_{XY}^2 = E[(X-\mu_X)(Y-\mu_Y)] = \sigma_{YX}^2$$

$$E[(X-\mu_X)(Y-\mu_Y)] = \sum (x-\mu_X)(y-\mu_Y)P(X=x, Y=y)$$

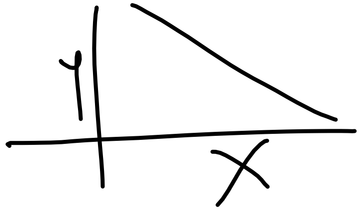
linear relationship betw two variable

Example

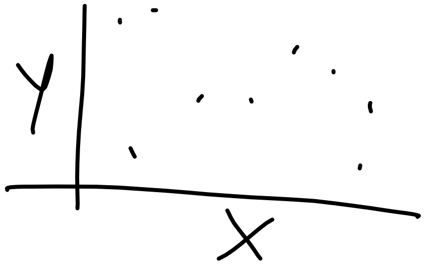
$\sigma_{xy}^2 \rightarrow +ve$ $\quad X\uparrow \; Y\uparrow \quad X\downarrow \; Y\downarrow$



$\sigma_{xy}^2 \longrightarrow -ve$ $\quad X\uparrow \; Y\downarrow \quad X\downarrow, \; Y\uparrow$



$\sigma_{xy}^2 = 0$ $\quad X\uparrow\downarrow \perp Y\downarrow\uparrow$



$X \rightarrow$

$Y \rightarrow$

1, ..., 6

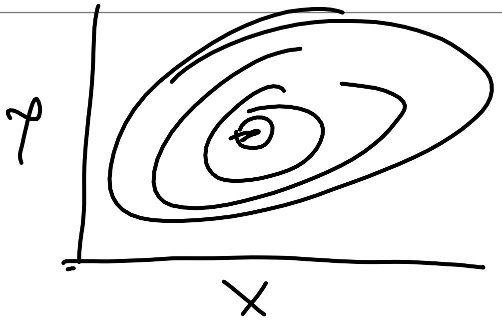
Dice

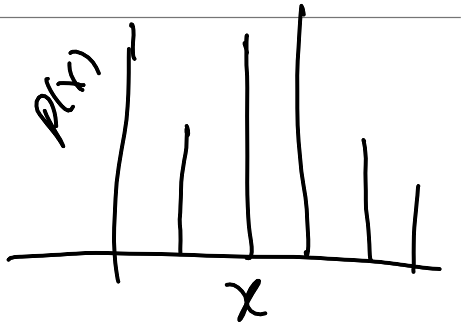
$X \rightarrow$ 1 4 ,

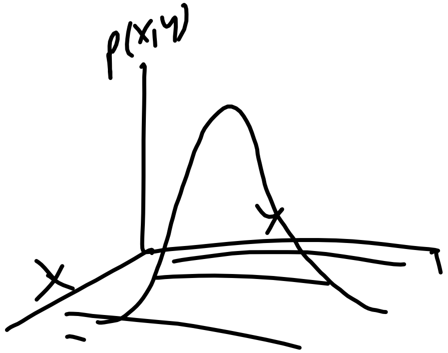
$Y \rightarrow$ 3 1

(1 , 6)

$\sigma^2_{XY} \approx 0$







$X \to 1, 2, 3, 4, 5, 6$

$Y \to 0, 0, 1, 0, 3, 4, 6$

Estimation of Covariance

Variance of x

$$\sigma_X^2 = E[(X-\mu_X)^2] = E[(X-\mu_X)(X-\mu_X)] = \sum (x-\mu_X)^2 P(X=x)$$

Covariance of x and y (cross-covariance)

$$\sigma_{XY}^2 = E[(X-\mu_X)(Y-\mu_Y)] = \sigma_{YX}^2$$

$$E[(X-\mu_X)(Y-\mu_Y)] = \sum (x-\mu_X)(y-\mu_Y)\boxed{P(X=x, Y=y)}$$

Covariance of x and y can be estimated as

$$\hat{\sigma}_{XY}^2 = Cov(x,y) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\hat{\mu}_X)(y_i-\hat{\mu}_Y)$$

Example

Correlation

Correlation is closely related to covariance, so it can be computed as:

ρ_{XY}

$X \rightarrow$

$Y \rightarrow$

$$\boxed{\rho_{XY} = \frac{\sigma^2_{XY}}{\sigma_X \sigma_Y}}$$

$$-1 \leq \rho_{XY} \leq 1$$

$$\boxed{Corr(x,y) = \frac{Cov(x,y)}{\hat{\sigma}_x \hat{\sigma}_y}} = \frac{\hat{\sigma}^2_{xy}}{\hat{\sigma}_x \hat{\sigma}_y}$$

Cor

$r_{xy} =$

$$= \frac{\sum(x-\bar{x})(y-\bar{y})}{\left(\sum(x-\bar{x})^2 \sum(y-\bar{y})^2\right)^{1/2}}$$

Example

Independently and Identically Distributed: IID

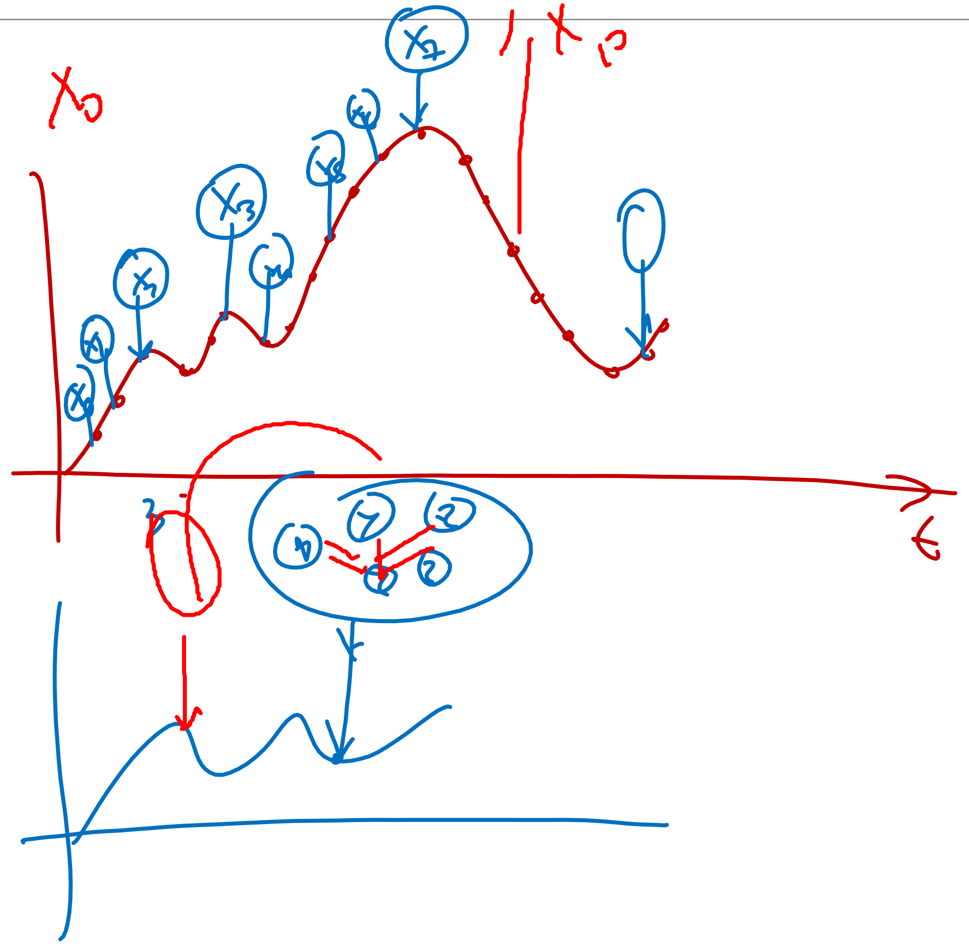
ii iid

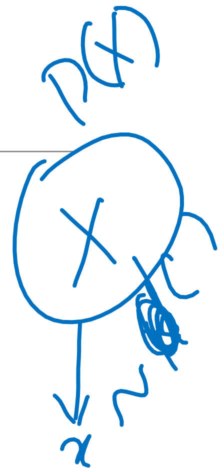
$$z_i \sim N(0, 1) \quad \text{iid}$$

Stochastic Processes: Time series

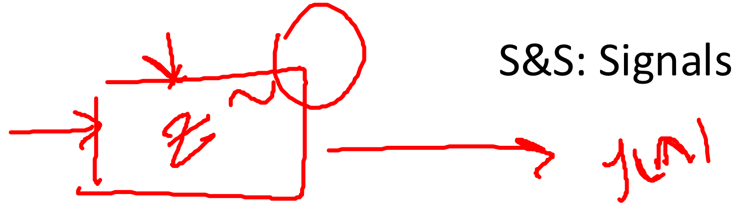
Collection of Random variable

- $X_1 \quad X_2 \quad X_3 \quad \dots$ $\qquad X_n \to x_n$
- 20 32 22
- $X_n \sim distribution(\mu_n, \sigma_n^2, \ \dots),$
- Time series $\qquad X_t \to x_t$

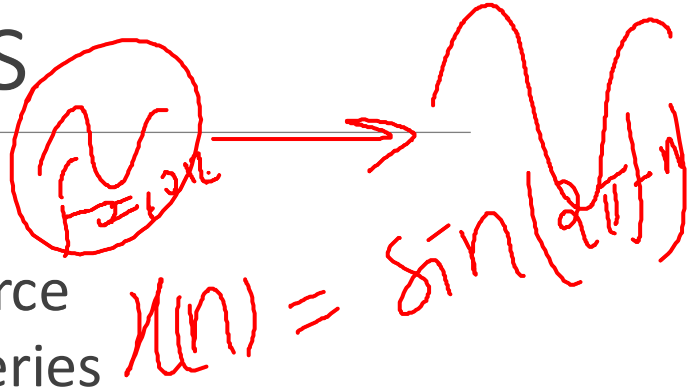




S&S: Signals & Systems

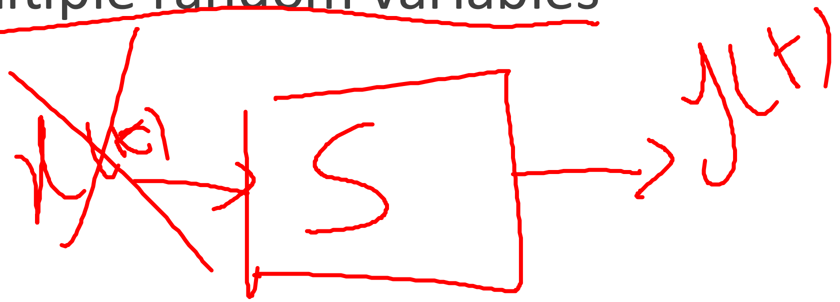


Stochastic Processes: Time series



Stochastic process is a collection of Random variable

- Opposite of deterministic process, deterministic system/source
- We are more interested to learn about the **source** of time-series
- S&S: A source can be seen as a system, with no or a little input
- S&S: A time-series is a signal, output of the system
- Every sample produced at time n, can be seen as a random variable.
- All the samples of a time-series generated from multiple random variables with each having different distribution



- $X_n \sim distribution(\mu_n, \sigma_n^2, \ \dots),$

Example

Types of characteristics

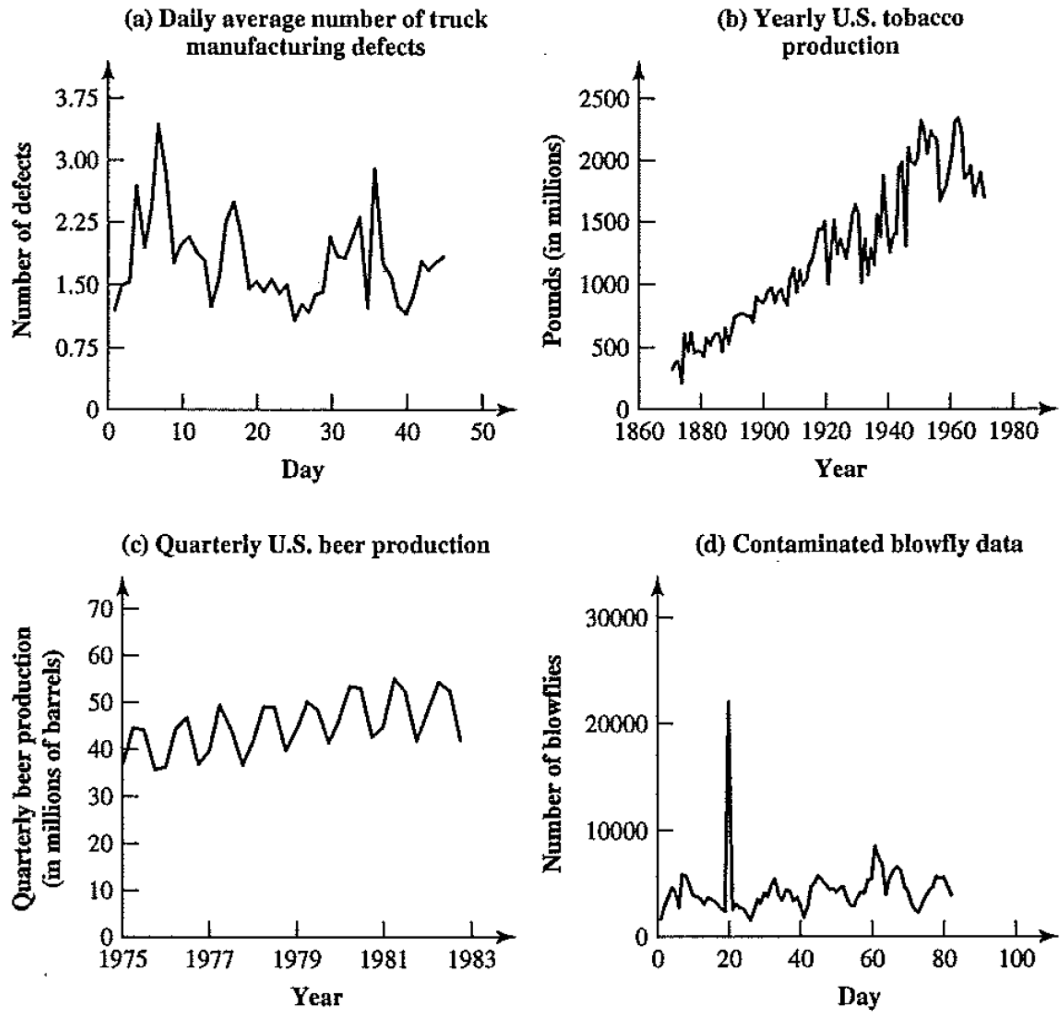
Time-series ≡ signal

(a) (weak) Stationary

(b) Non-stationary

(c) Season variation: seasonality

(d) Non-stationary due to external disturbance



Strict Stationary

detrend

trend

Stationarity

Stationary: plot (a):

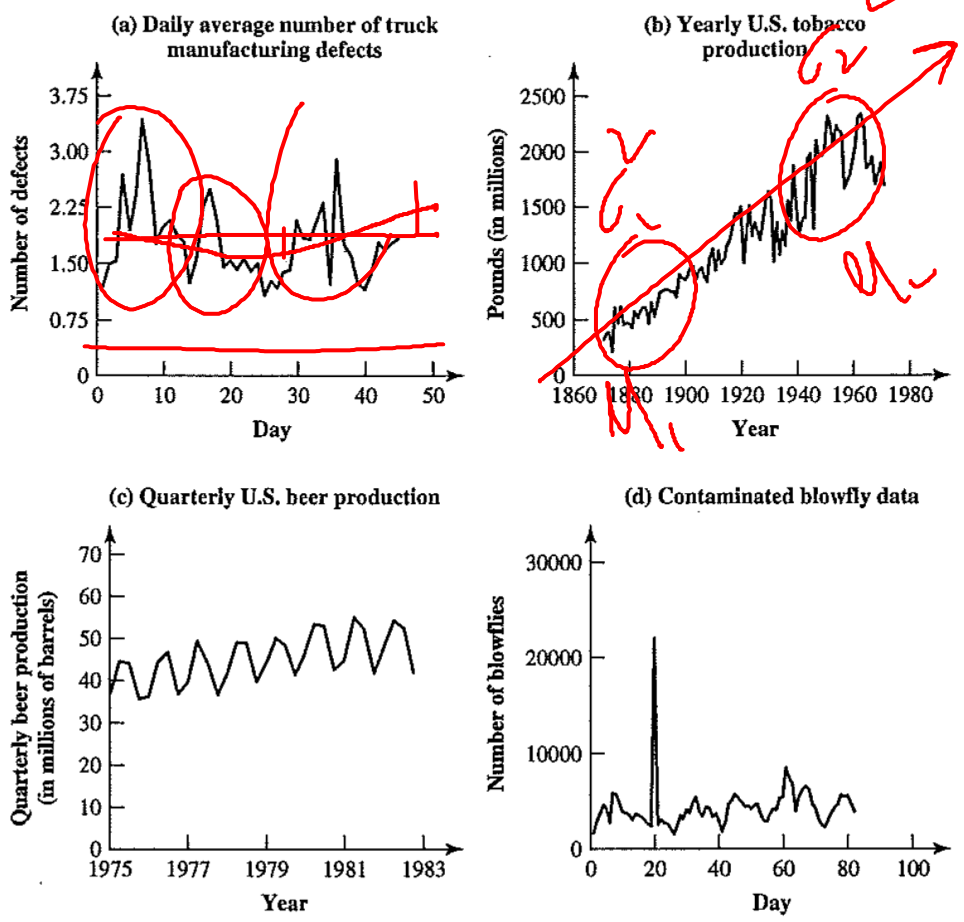
- No systematic change in time-series,
- No trends,
- No systematic change in mean
- No systematic change in variation
- No period fluctuation

μ

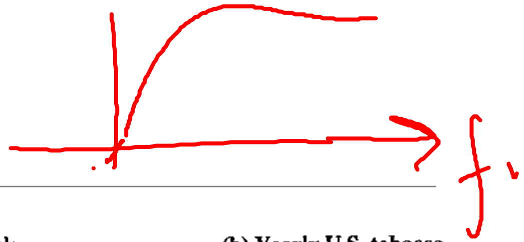
Properties of one part (section) of signal, is same as another part of signal

It is property of the stochastic process of the model, not the property of time-series, but often we say - a (weak) stationary time series

Why weak?



Direct current
DC filter
of freq component
filter



Non-Stationarity

Non-Stationary: plot (b), (d):

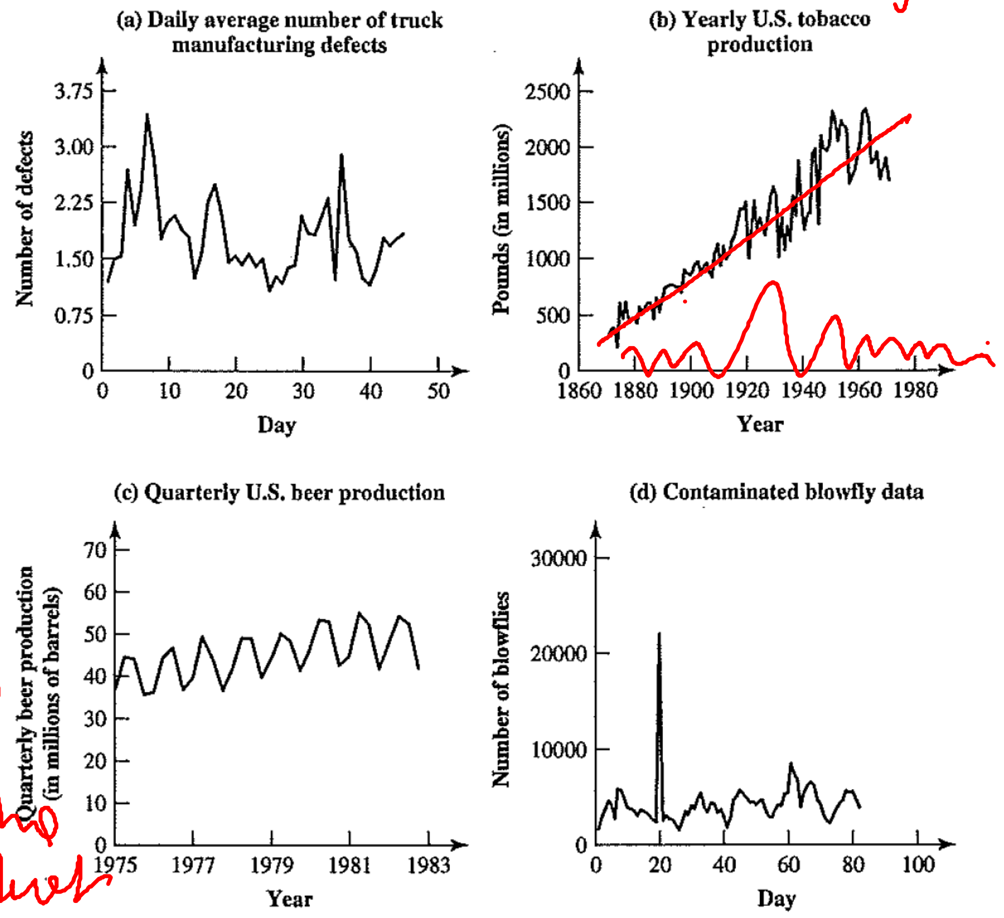
- trends,

*Properties of one part (section) of signal, is **NOT** same as another part of signal*

Transform into Stationary time series

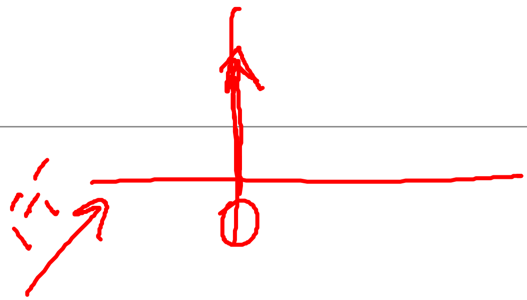
detrend

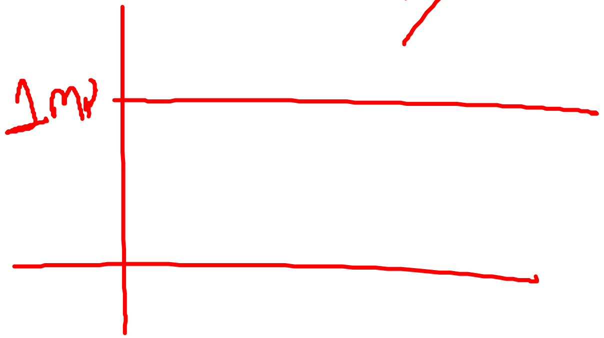
Removing trend
Removing Butterworth
DC Removal



Direct current

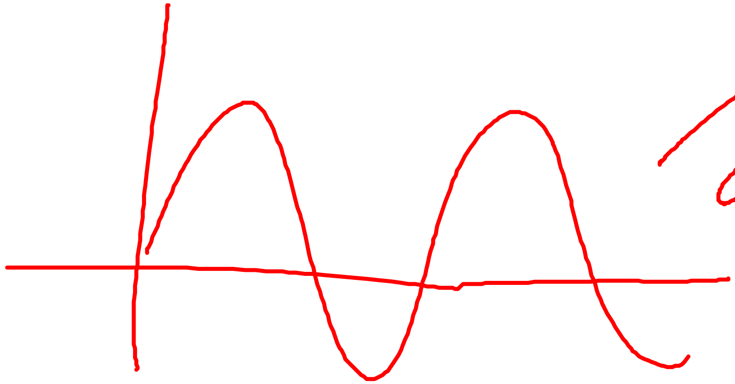
DC





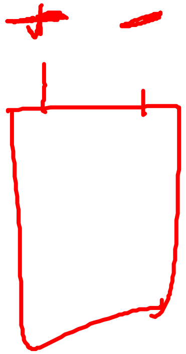
No

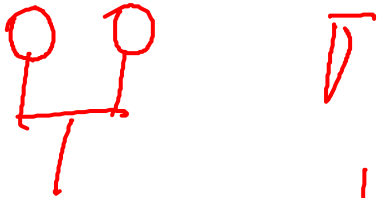
AC Alternating current

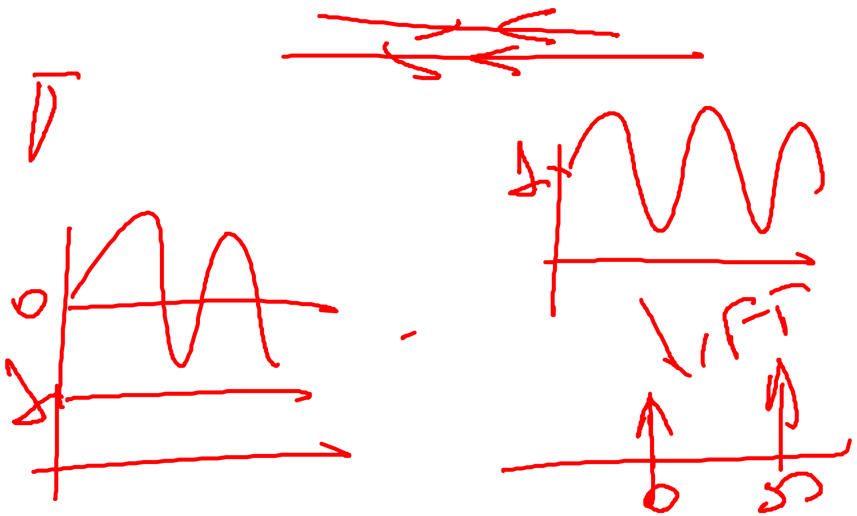


50Hz

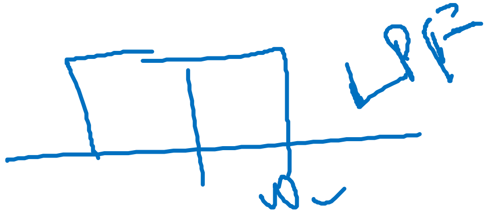
No =0





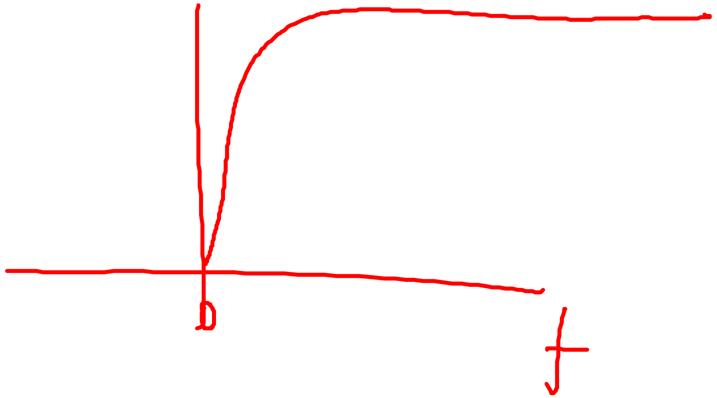


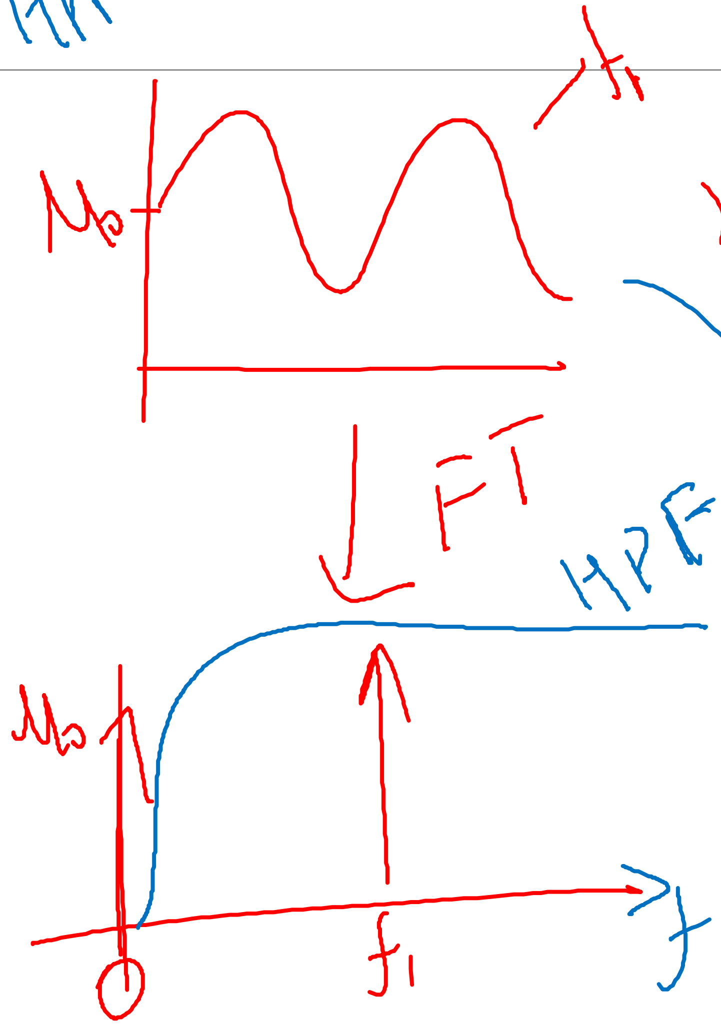
DC Removal

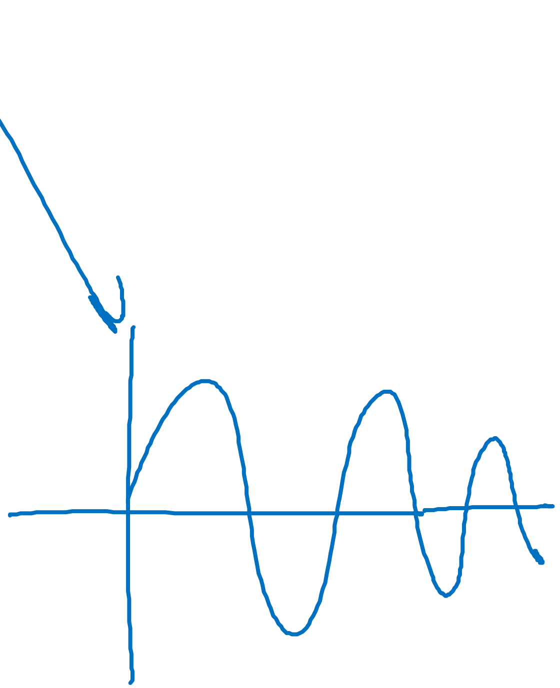


DC Filtering

$x(t) = M_0 + \sin(f_1 t)$







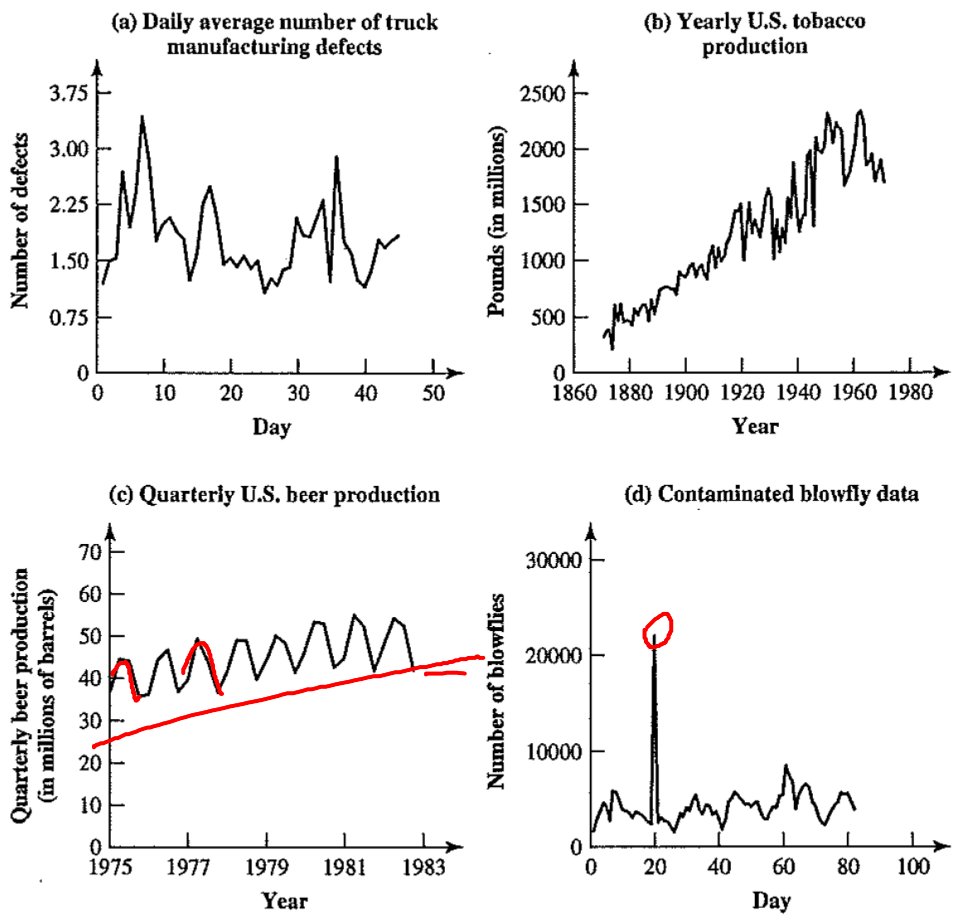
~*Periodicity*

Seasonality: Season variation:

Seasonality: plot (c):

- Periodic fluctuations,

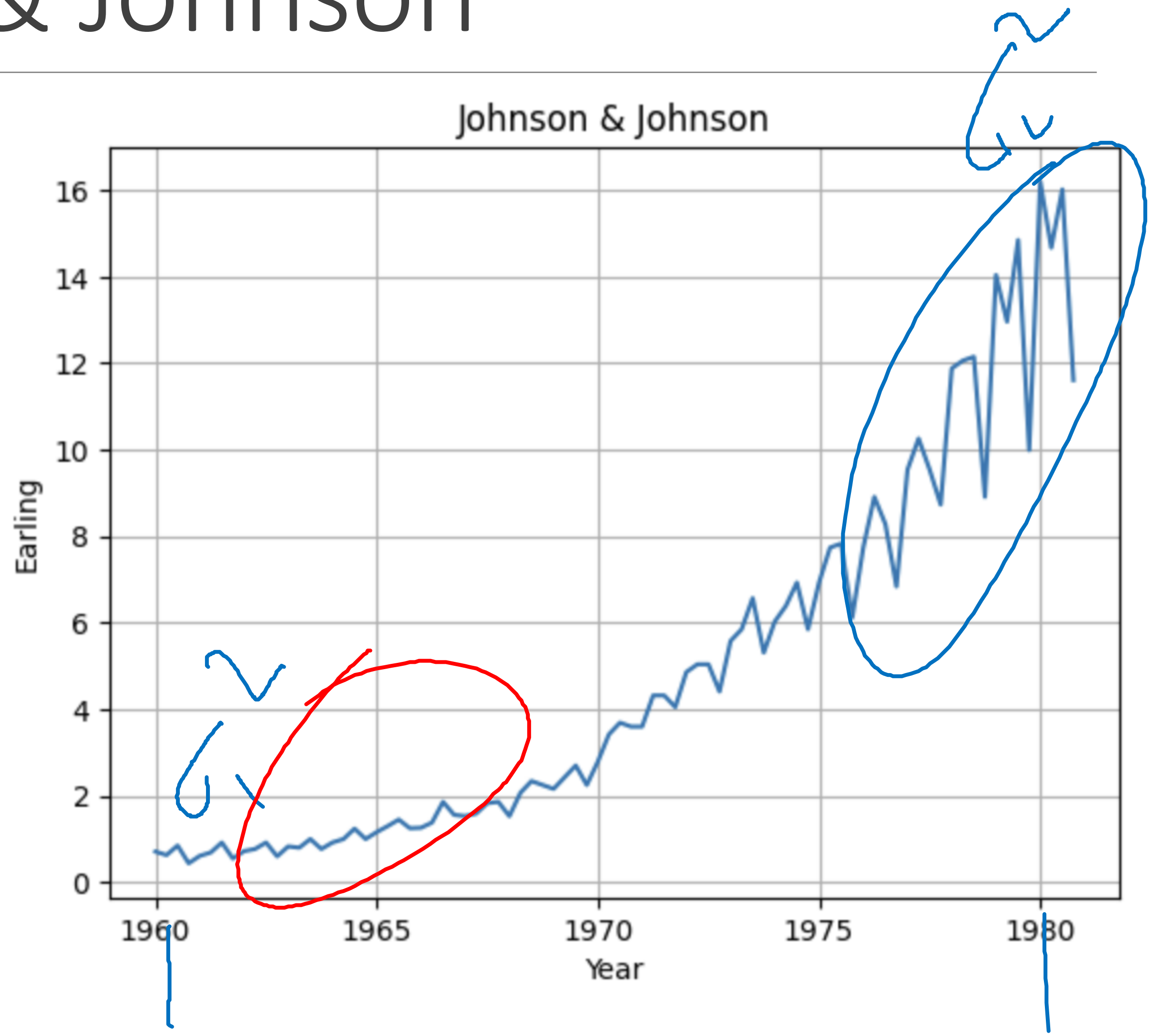
Time-series have periodicity, something that repeats like seasons



Example: Johnson & Johnson

Johnson & Johnson, quarterly Earning per share

- Collected from 21 years
- see anything meaningful?



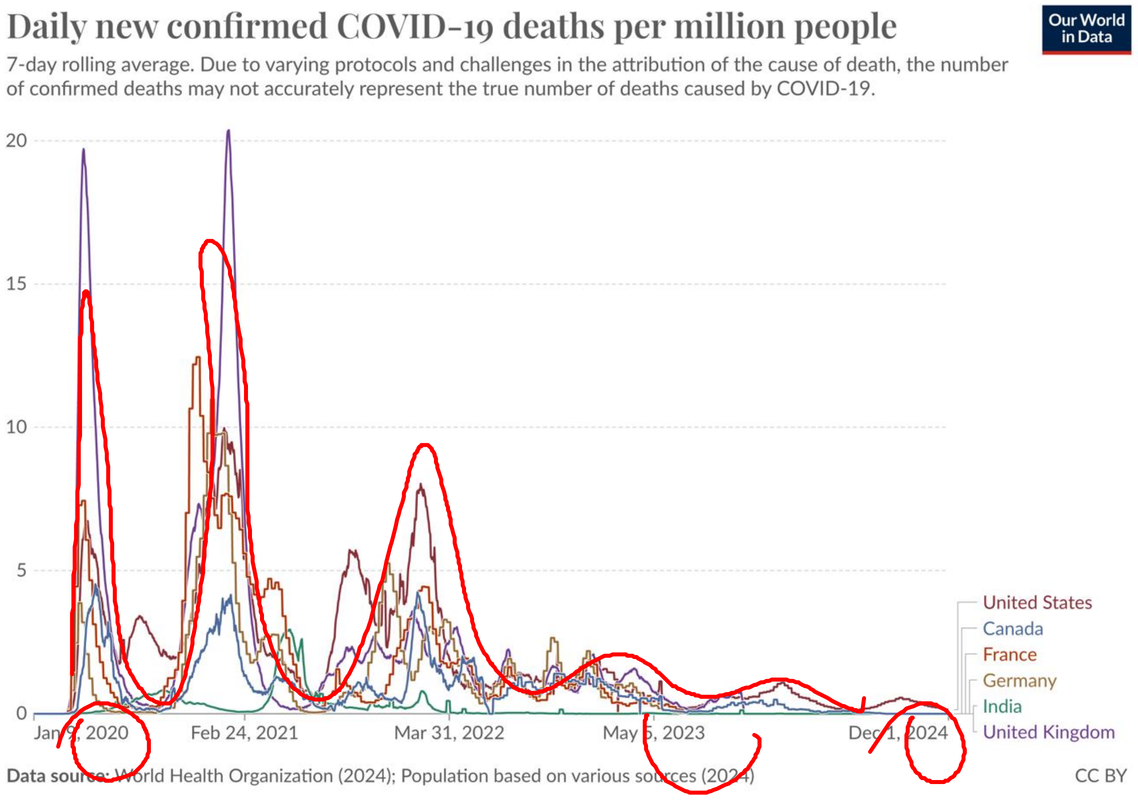
Example: Covid-19 Deaths

Covid-19 deaths per millions

- Check online

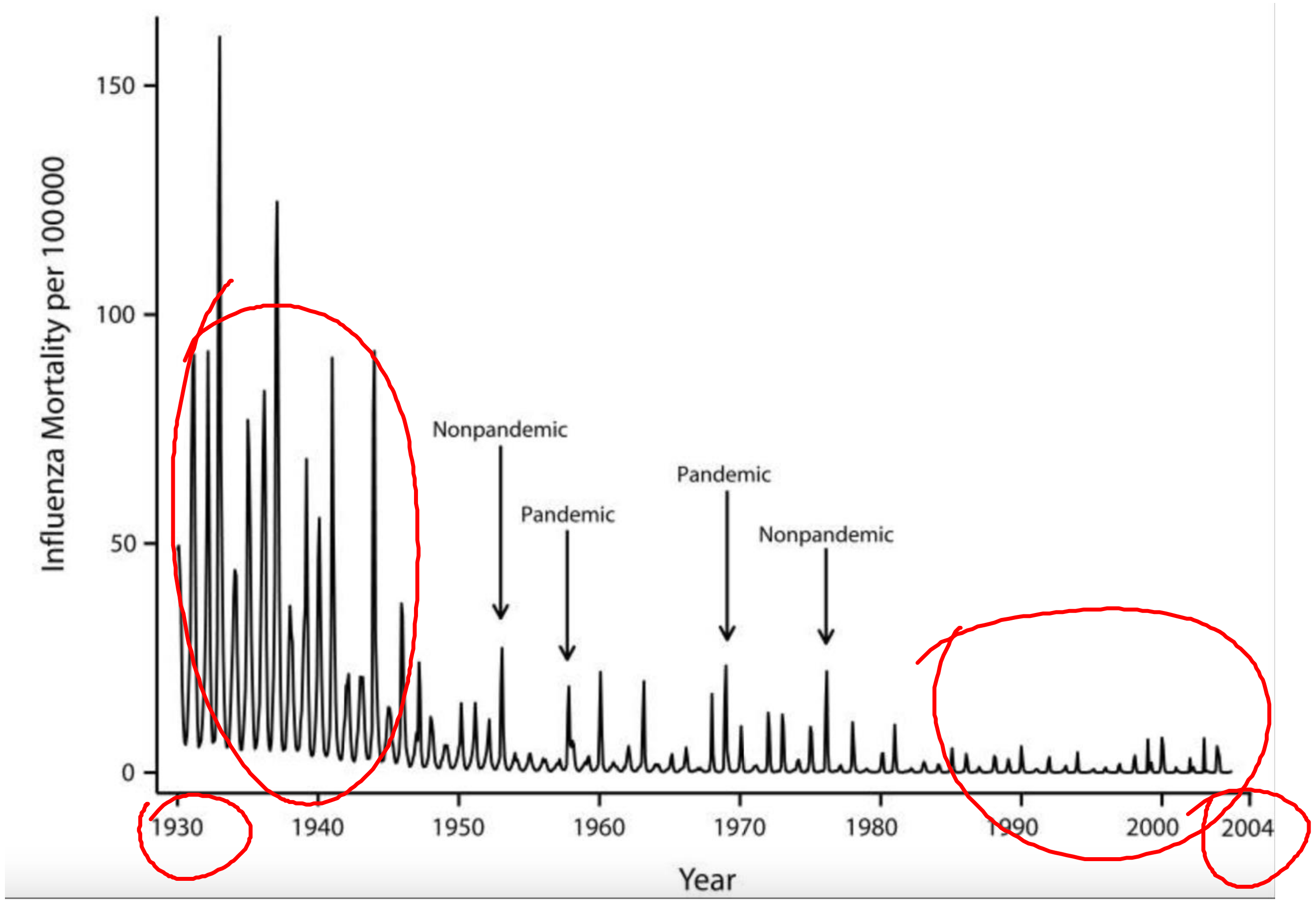
- see anything meaningful?

[Source: https://ourworldindata.org/covid-deaths](https://ourworldindata.org/covid-deaths)



Example: Influenza

- see anything meaningful?



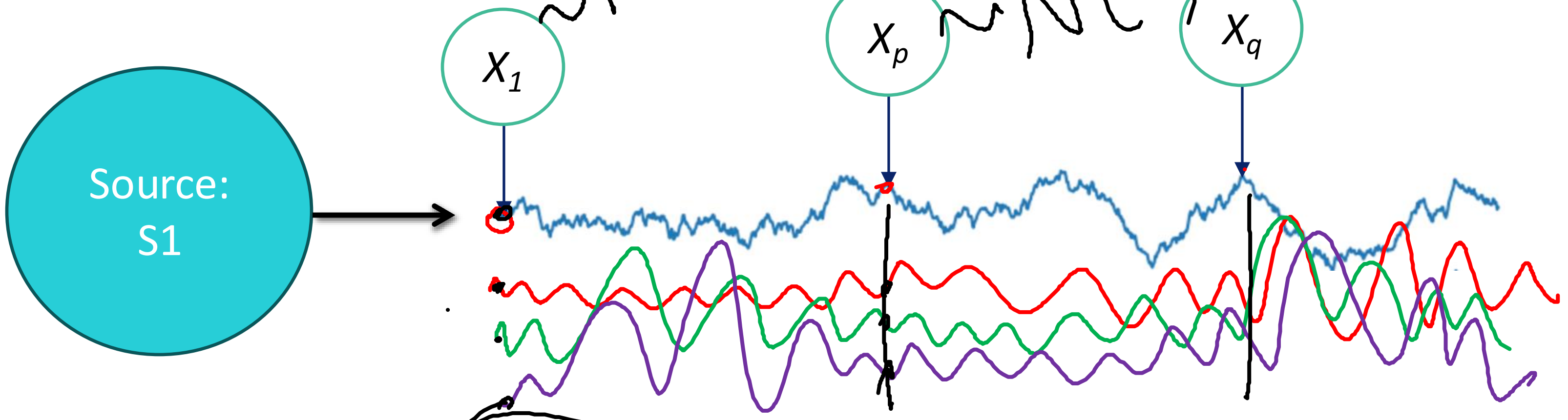
Source:
https://pmc.ncbi.nlm.nih.gov/articles/PMC2374803/

Autocovariance Function: AVF

Recall Covariance of two random variables X and Y: $\sigma^2_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sigma^2_{YX}$

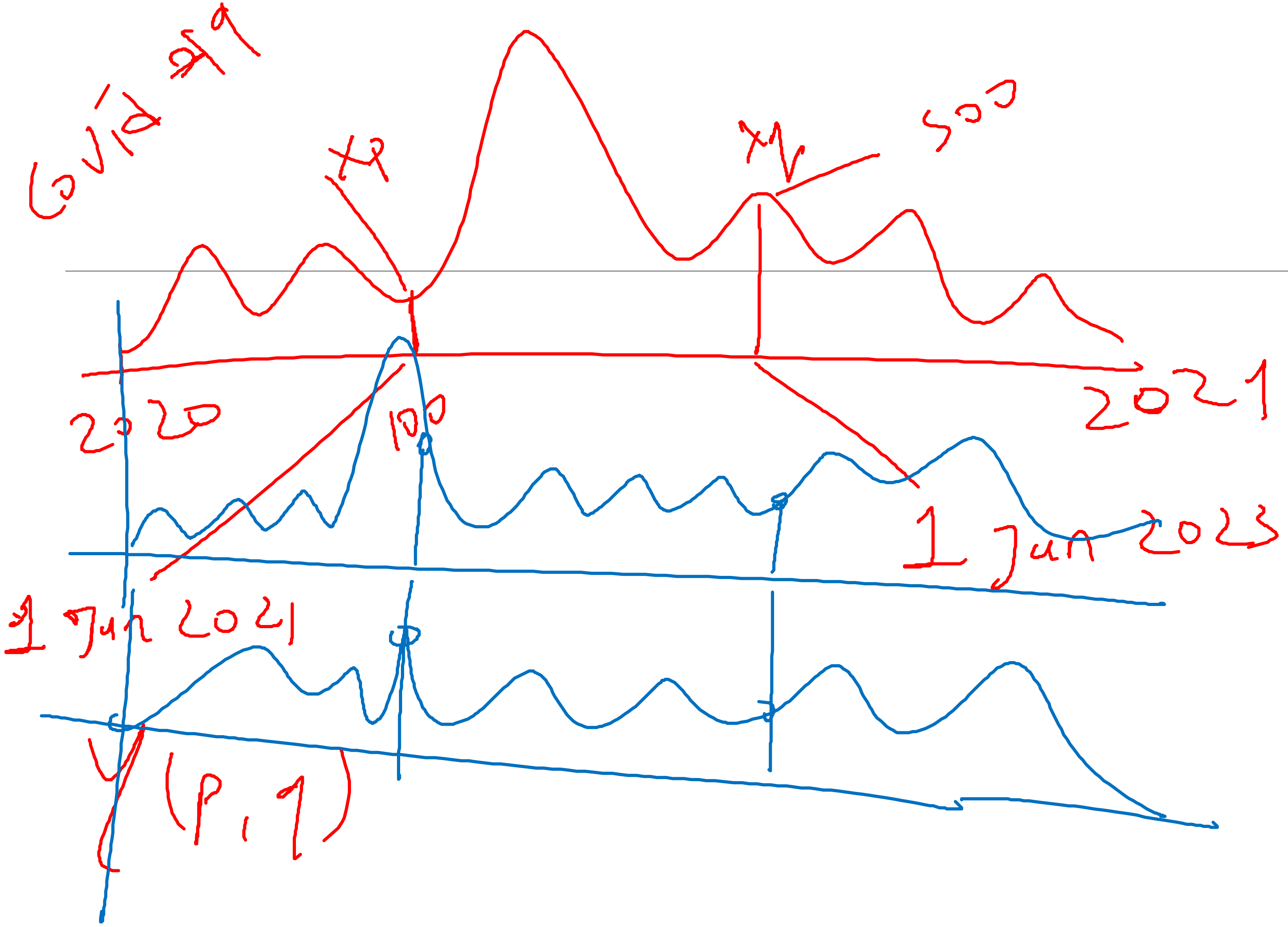
$\sim N(\ )$ $\sim N(\ )$

Time-series



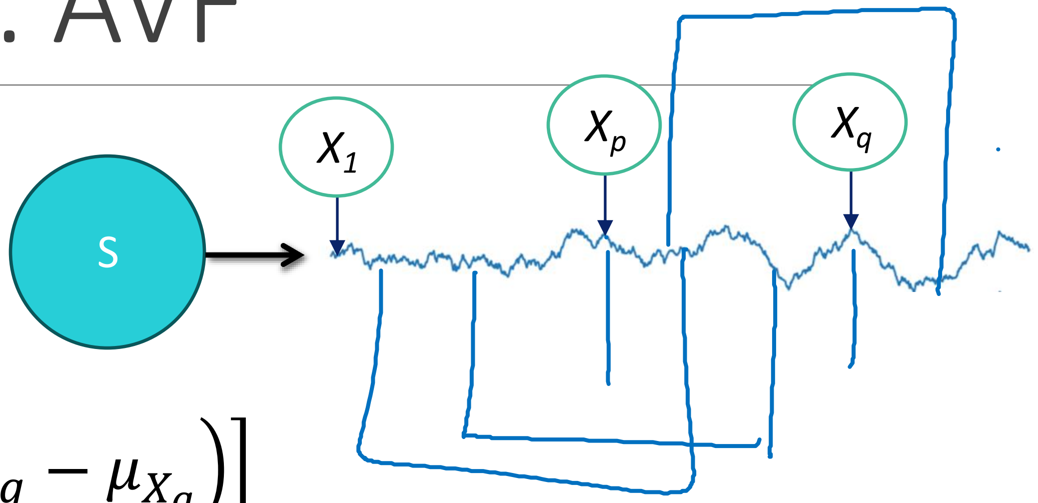
- Autocovariance Function: $\gamma(p,q)$

$$\gamma(p,q) = \sigma^2_{X_pX_q} = E\left[\left(X_p - \mu_{X_p}\right)\left(X_q - \mu_{X_q}\right)\right]$$


Covid 19
2020
2021
100
500
1 Jun 2023
1 Jun 2021
(P, 1)

Autocovariance Function: AVF

- Autocovariance Function: $\gamma(p, q)$



$$\gamma(p, q) = \sigma^2_{X_p X_q} = E\left[\left(X_p - \mu_{X_p}\right)\left(X_q - \mu_{X_q}\right)\right]$$

$$\gamma(p, p) = \sigma^2_{X_p} = E\left[\left(X_p - \mu_{X_p}\right)^2\right]$$

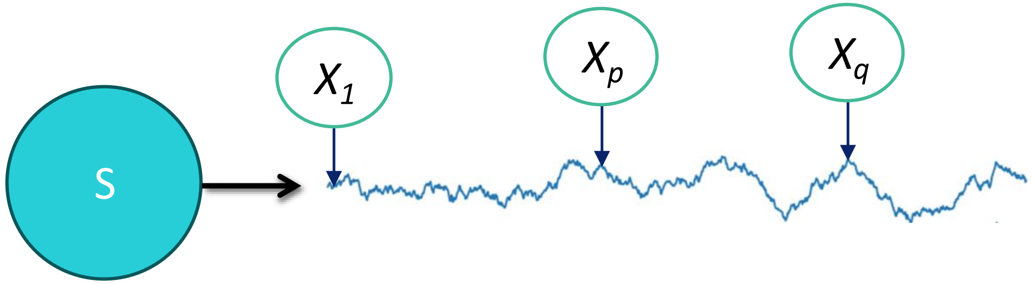
- Assuming the stationary time series: *k - time difference*

$$\boxed{\gamma_k = \gamma(p, p+k) = \sigma^2_{X_p X_{p+k}}}$$

Autocovariance Function: AVF

AVF

$$\boxed{\gamma_k = \gamma(p, p+k) = \sigma^2_{X_p X_{p+k}}}$$



γ_2

X_0

X

$P: 1, 2, 3, 4, 5, 1, 1, 1, 7, 8, 2$

X_2

γ_3 X_0

X_3

$$\hat{\sigma}^2_{XY} = Cov(x, y) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu}_X)(y_i - \hat{\mu}_Y)$$

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Autocovariance Coefficients: γ_k

Autocovariance function for a stationary time-series

$$\gamma_k = \gamma(p, p+k) = \sigma^2_{X_p X_{p+k}} = E\left[\left(X_p - \mu_{X_p}\right)\left(X_{p+k} - \mu_{X_{p+k}}\right)\right]$$

Autocovariance Coefficients $\hat{\gamma}_k$ is estimation of γ_k

For given a realization x (time-series) of process

$$\boxed{\hat{\gamma}_k = \frac{1}{N-k}\sum_{i=1}^{N-k}(x_i - \bar{x})(x_{i+k} - \bar{x})}$$

$$\boxed{\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i}$$

Autocovariance Coefficients: c_k

$$\hat{\gamma}_k = \frac{1}{N} \sum_{i=1}^{N-k} (x_i - \bar{x})(x_{i+k} - \bar{x})$$

bias

bias

$$\hat{\hat{\gamma}}_k = \frac{1}{N - k} \sum_{i=1}^{N-k} (x_i - \bar{x})(x_{i+k} - \bar{x}) \quad = c_k$$

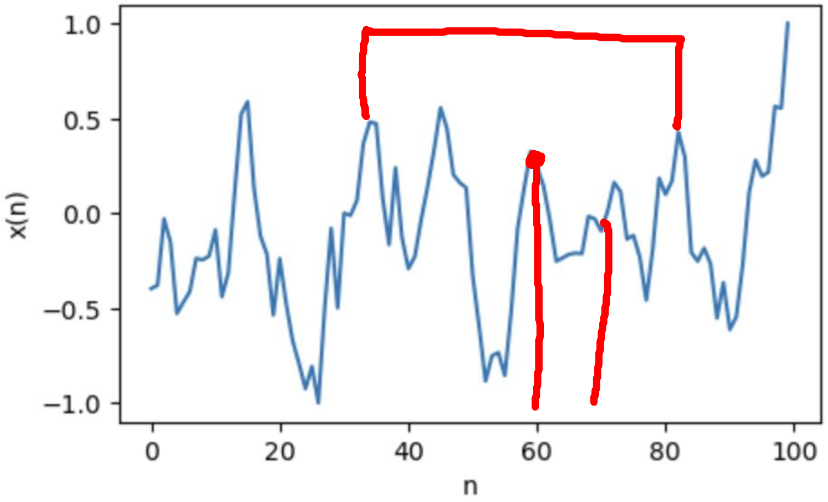
In general, $\hat{\gamma}_k$ has larger bias than $\hat{\hat{\gamma}}_k$

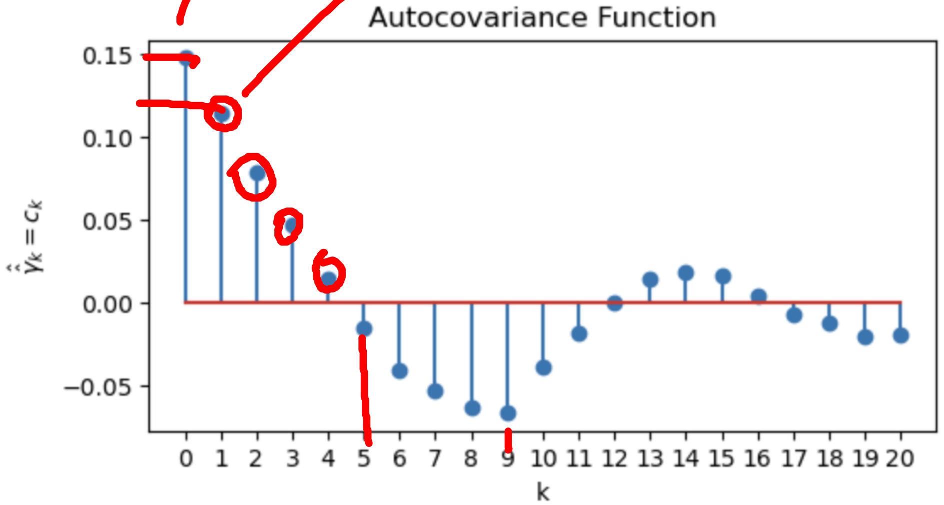
{proof: Book: 2.5.2}

Example: AVF

y_0 x_0

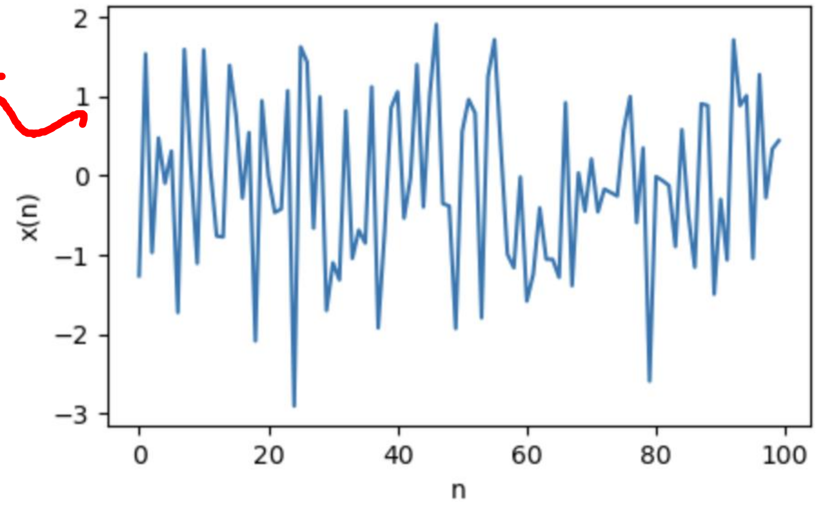
$y_1 \sim x_{0+1}$

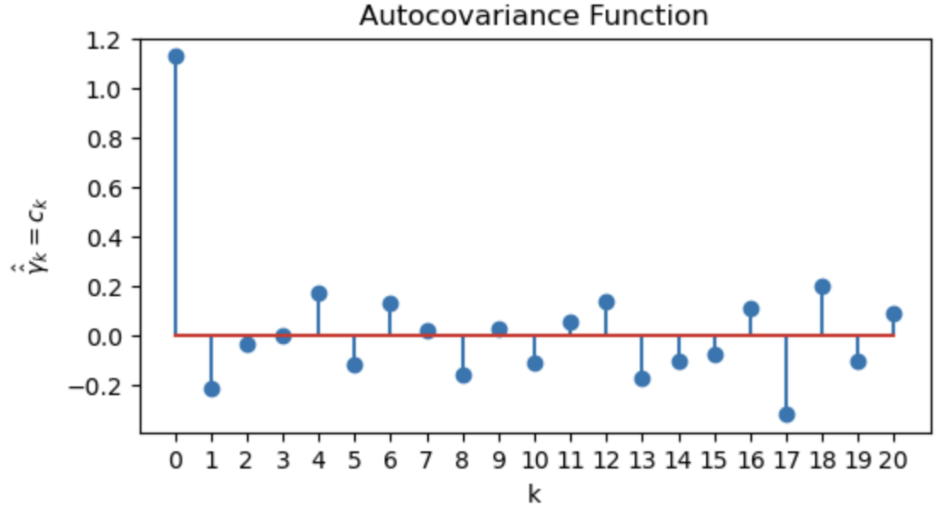




iid

$N(0,1)$





Example

Example

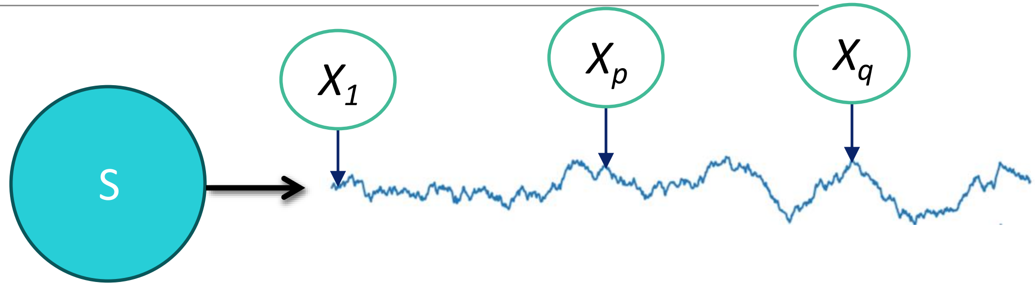
$$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$$

Autocorrelation Function: ACF

Autocorrelation Function: ρ_k

- Assume weak stationarity
- Computed for X_p and X_{p+k}
-

$$-1 \le \rho_k = \frac{\gamma_k}{\gamma_0} \le 1$$



- Estimation: *lag k*

$$\boxed{\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_0} = r_k}$$

$$\hat{\rho}_k = \frac{\hat{\hat{\gamma}}_k}{\hat{\hat{\gamma}}_0} = r_k$$

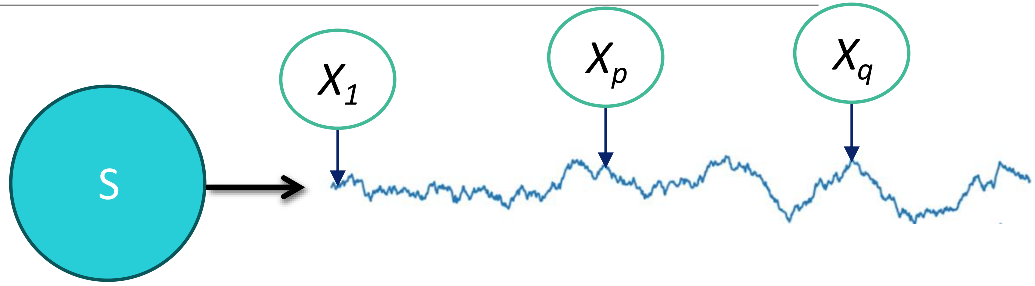
$$\hat{\rho}_0 = r_0 = 1$$

$$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$$

Autocorrelation Function: ACF

Autocorrelation Function: ρ_k

- Estimation: *lag k*



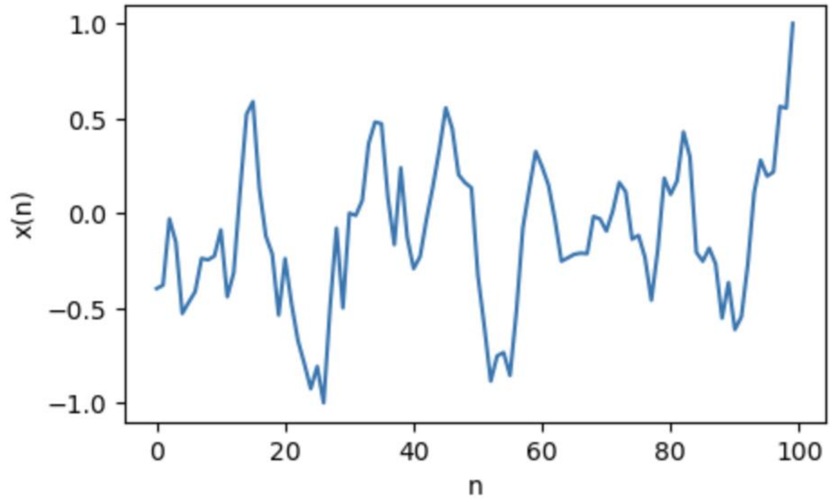
$$\hat{\rho}_k = \frac{\sum_{i=1}^{N-k}(x_i - \bar{x})(x_{i+k} - \bar{x})}{\sum_{i=1}^{N}(x_i - \bar{x})^2} = r_k$$

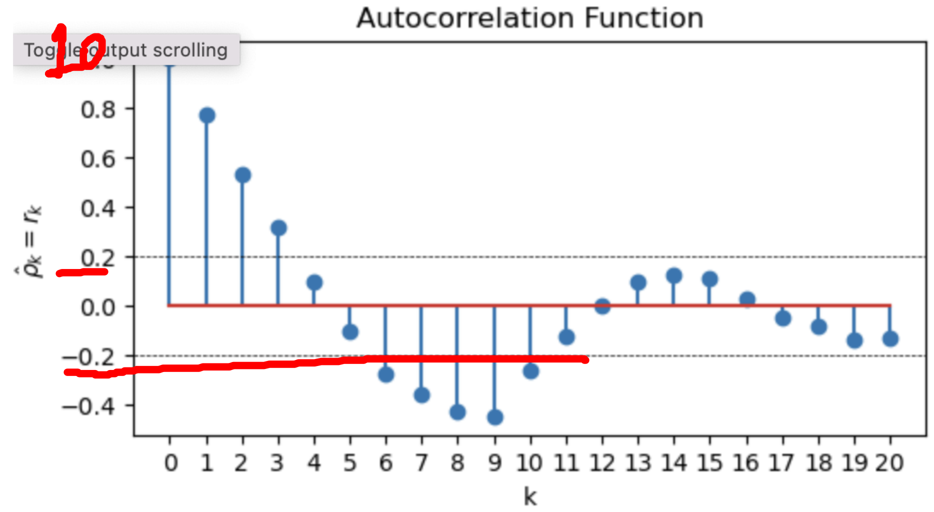
ρ rho

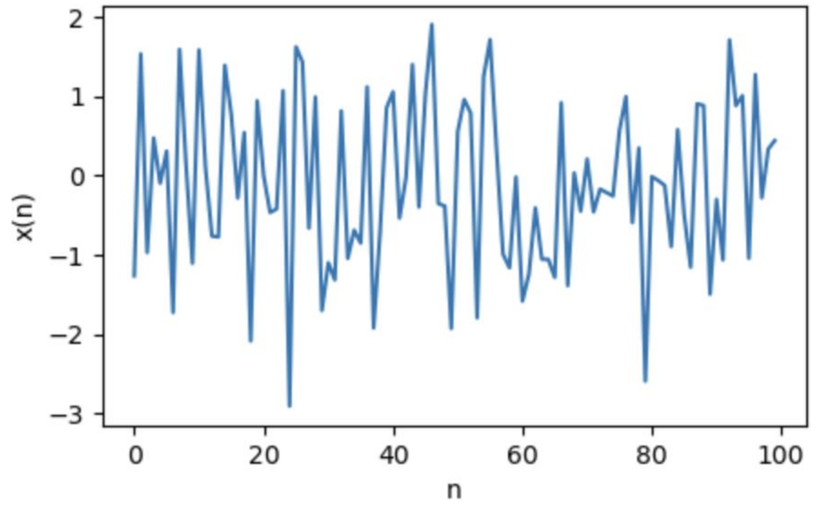
$$\hat{\rho}_0 = r_0 = 1$$

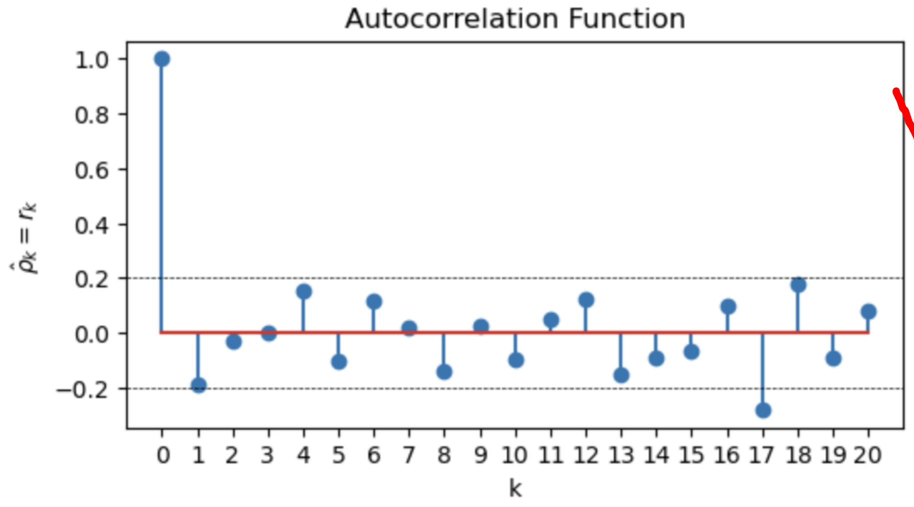
Example: ACF

<u>Correlogram</u>









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Example

Summary so far

$x(n) = [1\ 0\ 2\ 1\ 4\ 1\ 3\ 4\ 5\ 1\ 7\ 8]$

$\hat{\rho}_k = ?$

$\hat{\gamma}_k = ?$

So far, we have learned:

- Basic concepts of Stochastic Processes, Random Variables
- Time series as Stochastic Process
- Characteristics of time series: Stationarity, Seasonality, Non-stationarity

- Autocovariance Function AVF: γ_k and estimations
- Autocorrelation Function ACF: ρ_k and estimations

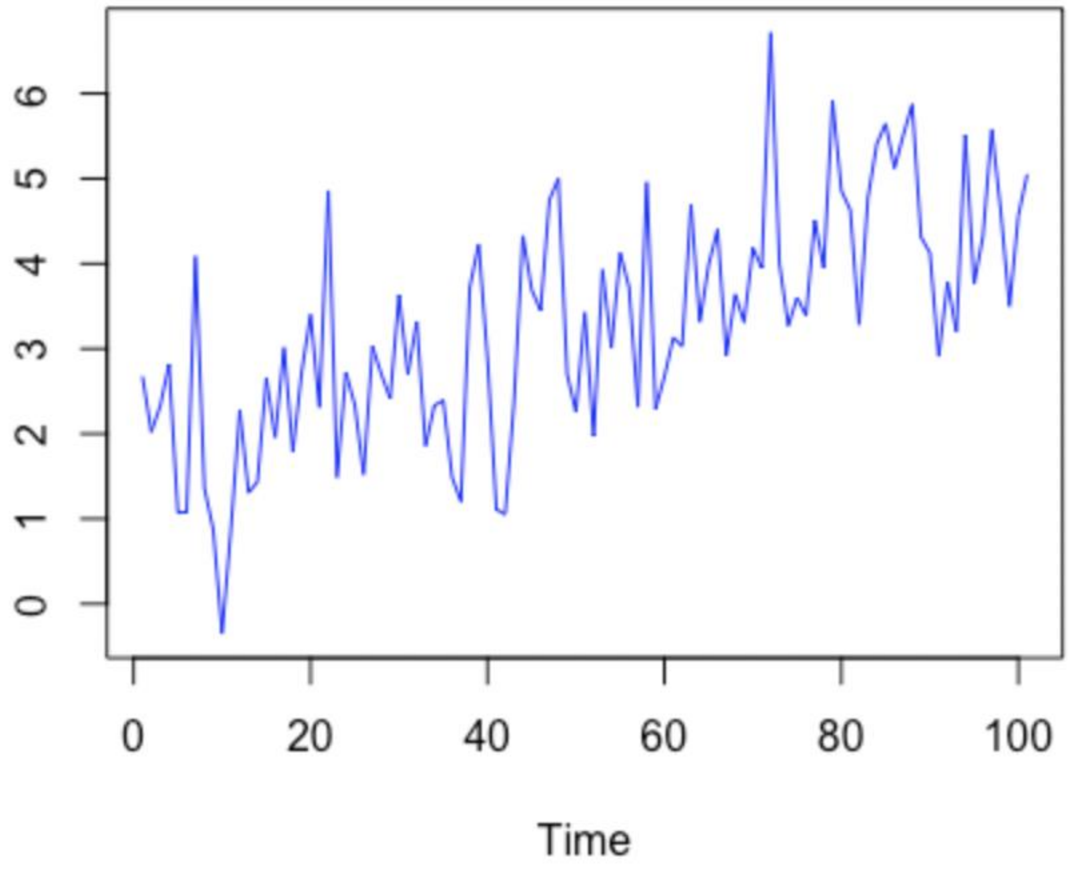
γ_5

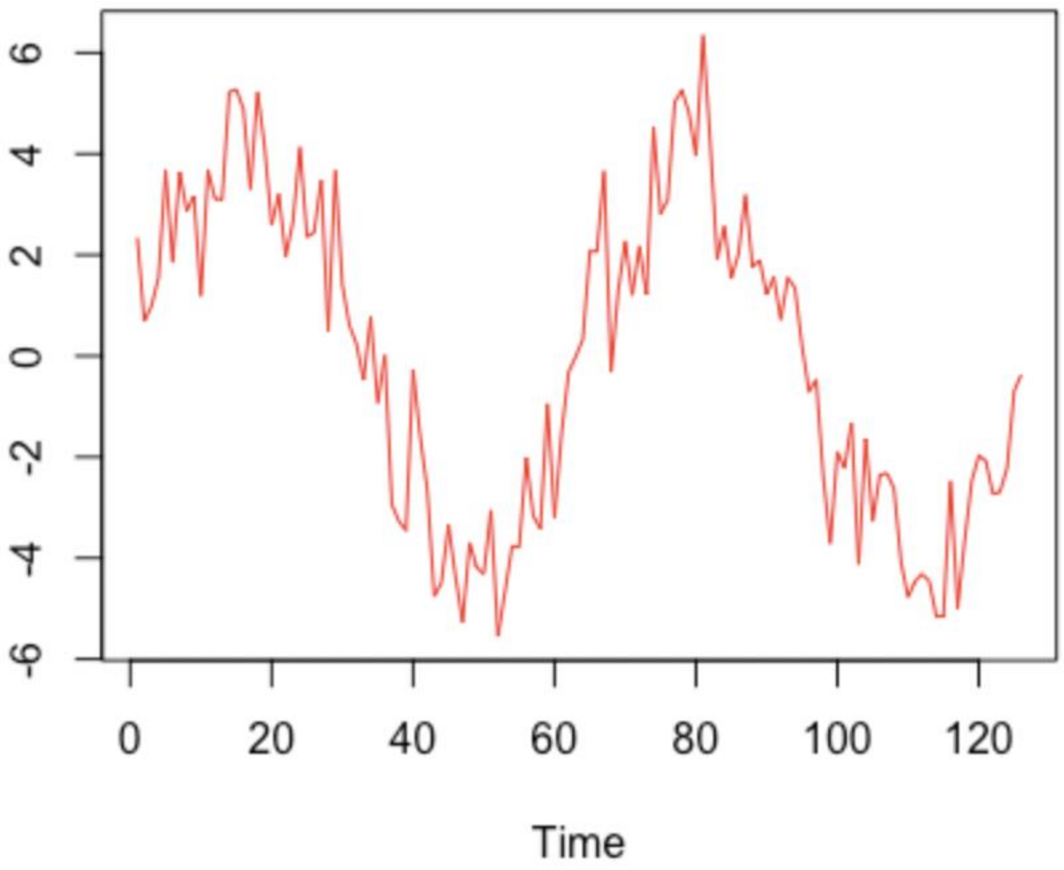
ρ_4

$1 \le k \le 9$

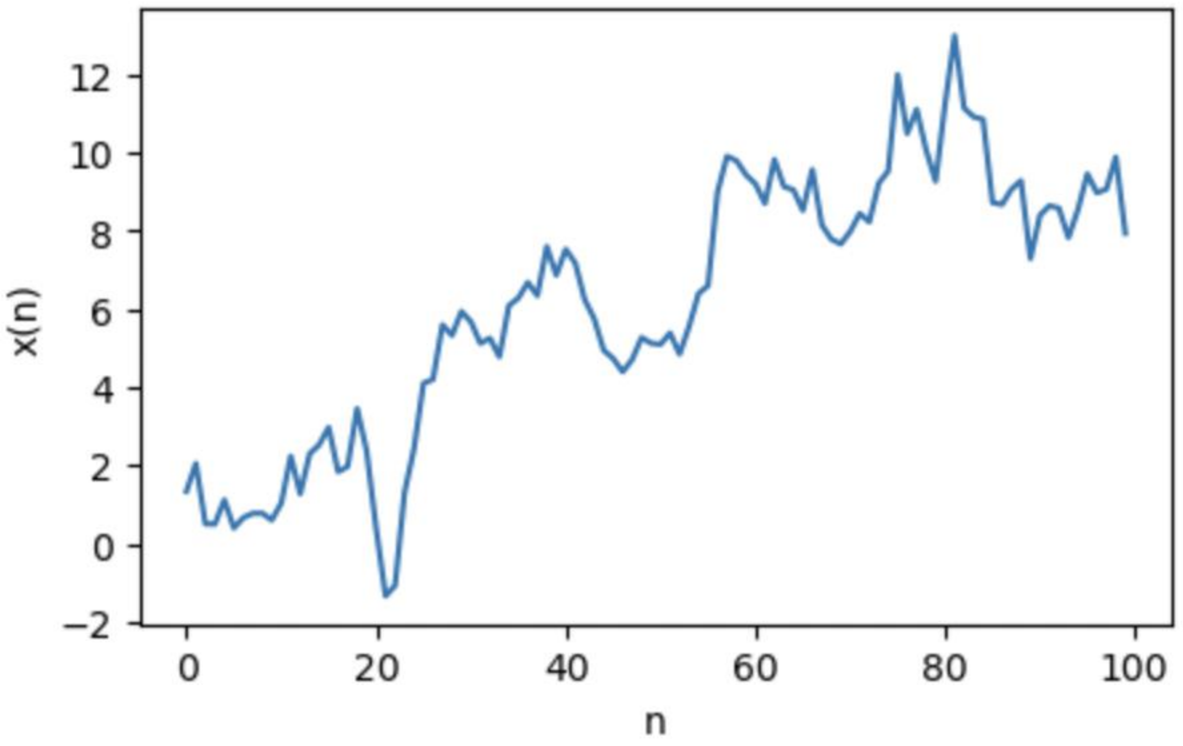
- Partial Autocorrelation Function PACF - will be covered after models

Let’s see some examples





6
4
2
0
-2
-4
-6
0
20
40
60
80
100
120
Time


x(n)
n
12
10
8
6
4
2
0
−2
0
20
40
60
80
100

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University of London